ON THE NATURE OF MATHEMATICAL TRUTH

C. G. HEMPEL, Queens College

1. The problem. It is a basic principle of scientific inquiry that no proposition and no theory is to be accepted without adequate grounds. In empirical science, which includes both the natural and the social sciences, the grounds for the acceptance of a theory consist in the agreement of predictions based on the theory with empirical evidence obtained either by experiment or by systematic observation. But what are the grounds which sanction the acceptance of mathematics? That is the question I propose to discuss in the present paper. For reasons which will become clear subsequently, I shall use the term "mathematics" here to refer to arithmetic, algebra, and analysis—to the exclusion, in particular, of geometry [1].

2. Are the propositions of mathematics self-evident truths? One of the several answers which have been given to our problem asserts that the truths of mathematics, in contradistinction to the hypotheses of empirical science, require neither factual evidence nor any other justification because they are "self-evident." This view, however, which ultimately relegates decisions as to mathematical truth to a feeling of self-evidence, encounters various difficulties. First of all, many mathematical theorems are so hard to establish that even to the specialist in the particular field they appear as anything but self-evident. Secondly, it is well known that some of the most interesting results of mathematics—especially in such fields as abstract set theory and topology—run counter to deeply ingrained intuitions and the customary kind of feeling of self-evidence. Thirdly, the existence of mathematical conjectures such as those of Goldbach and of Fermat, which are quite elementary in content and yet undecided up to this day, certainly shows that not all mathematical truths can be self-evident. And finally, even if self-evidence were attributed only to the basic postulates of mathematics, from which all other mathematical propositions can be deduced, it would be pertinent to remark that judgments as to what may be considered as self-evident are subjective; they may vary from person to person and certainly cannot constitute an adequate basis for decisions as to the objective validity of mathematical propositions.

3. Is mathematics the most general empirical science? According to another view, advocated especially by John Stuart Mill, mathematics is itself an empirical science which differs from the other branches such as astronomy, physics, chemistry, *etc.*, mainly in two respects: its subject matter is more general than that of any other field of scientific research, and its propositions have been tested and confirmed to a greater extent than those of even the most firmly established sections of astronomy or physics. Indeed, according to this view, the degree to which the laws of mathematics have been borne out by the past experiences of mankind is so overwhelming that—unjustifiably—we have come to think of mathematical theorems as qualitatively different from the well-confirmed hypotheses or theories of other branches of science: we consider them as

certain, while other theories are thought of as at best "very probable" or very highly confirmed.

But this view, too, is open to serious objections. From a hypothesis which is empirical in character—such as, for example, Newton's law of gravitation—it is possible to derive predictions to the effect that under certain specified conditions certain specified observable phenomena will occur. The actual occurrence of these phenomena constitutes confirming evidence, their non-occurrence disconfirming evidence for the hypothesis. It follows in particular that an empirical hypothesis is theoretically disconfirmable; *i.e.*, it is possible to indicate what kind of evidence, if actually encountered, would disconfirm the hypothesis. In the light of this remark, consider now a simple "hypothesis" from arithmetic: $3+2=5$. If this is actually an empirical generalization of past experiences, then it must be possible to state what kind of evidence would oblige us to concede the hypothesis was not generally true after all. If any disconfirming evidence for the given proposition can be thought of, the following illustration might well be typical of it: We place some microbes on a slide, putting down first three of them and then another two. Afterwards we count all the microbes to test whether in this instance 3 and 2 actually added up to 5. Suppose now that we counted 6 microbes altogether. Would we consider this as an empirical disconfirmation of the given proposition, or at least as a proof that it does not apply to microbes? Clearly not; rather, we would assume we had made a mistake in counting or that one of the microbes had split in two between the first and the second count. But under no circumstances could the phenomenon just described invalidate the arithmetical proposition in question; for the latter asserts nothing whatever about the behavior of microbes; it merely states that any set consisting of $3+2$ objects may also be said to consist of 5 objects. And this is so because the symbols "$3+2$" and "5" denote the same number: they are synonymous by virtue of the fact that the symbols "2," "3," "5," and "$+$" are *defined* (or tacitly understood) in such a way that the above identity holds as a consequence of the meaning attached to the concepts involved in it.

4. The analytic character of mathematical propositions. The statement that $3+2=5$, then, is true for similar reasons as, say, the assertion that no sexagenarian is 45 years of age. Both are true simply by virtue of definitions or of similar stipulations which determine the meaning of the key terms involved. Statements of this kind share certain important characteristics: Their validation naturally requires no empirical evidence; they can be shown to be true by a mere analysis of the meaning attached to the terms which occur in them. In the language of logic, sentences of this kind are called analytic or true a priori, which is to indicate that their truth is logically independent of, or logically prior to, any experiential evidence [2]. And while the statements of empirical science, which are synthetic and can be validated only a posteriori, are constantly subject to revision in the light of new evidence, the truth of an analytic statement can be established definitely, once and for all. However, this characteristic "theoretical

certainty" of analytic propositions has to be paid for at a high price: An analytic statement conveys no factual information. Our statement about sexagenarians, for example, asserts nothing that could possibly conflict with any factual evidence: it has no factual implications, no empirical content; and it is precisely for this reason that the statement can be validated without recourse to empirical evidence.

Let us illustrate this view of the nature of mathematical propositions by reference to another, frequently cited, example of a mathematical—or rather logical—truth, namely the proposition that whenever $a=b$ and $b=c$ then $a=c$. On what grounds can this so-called "transitivity of identity" be asserted? Is it of an empirical nature and hence at least theoretically disconfirmable by empirical evidence? Suppose, for example, that a, b, c, are certain shades of green, and that as far as we can see, $a=b$ and $b=c$, but clearly $a\neq c$. This phenomenon actually occurs under certain conditions; do we consider it as disconfirming evidence for the proposition under consideration? Undoubtedly not; we would argue that if $a\neq c$, it is impossible that $a=b$ and also $b=c$; between the terms of at least one of these latter pairs, there must obtain a difference, though perhaps only a subliminal one. And we would dismiss the possibility of empirical disconfirmation, and indeed the idea that an empirical test should be relevant here, on the grounds that identity is a transitive relation by virtue of its definition or by virtue of the basic postulates governing it [3]. Hence, the principle in question is true a priori.

5. Mathematics as an axiomatized deductive system. I have argued so far that the validity of mathematics rests neither on its alleged self-evidential character nor on any empirical basis, but derives from the stipulations which determine the meaning of the mathematical concepts, and that the propositions of mathematics are therefore essentially "true by definition." This latter statement, however, is obviously oversimplified and needs restatement and a more careful justification.

For the rigorous development of a mathematical theory proceeds not simply from a set of definitions but rather from a set of non-definitional propositions which are not proved within the theory; these are the postulates or axioms of the theory [4]. They are formulated in terms of certain basic or primitive concepts for which no definitions are provided within the theory. It is sometimes asserted that the postulates themselves represent "implicit definitions" of the primitive terms. Such a characterization of the postulates, however, is misleading. For while the postulates do limit, in a specific sense, the meanings that can possibly be ascribed to the primitives, any self-consistent postulate system admits, nevertheless, many different interpretations of the primitive terms (this will soon be illustrated), whereas a set of definitions in the strict sense of the word determines the meanings of the definienda in a unique fashion.

Once the primitive terms and the postulates have been laid down, the entire theory is completely determined; it is derivable from its postulational basis in

the following sense: Every term of the theory is definable in terms of the primitives, and every proposition of the theory is logically deducible from the postulates. To be entirely precise, it is necessary also to specify the principles of logic which are to be used in the proof of the propositions, *i.e.* in their deduction from the postulates. These principles can be stated quite explicitly. They fall into two groups: Primitive sentences, or postulates, of logic (such as: If p and q is the case, then p is the case), and rules of deduction or inference (including, for example, the familiar modus ponens rule and the rules of substitution which make it possible to infer, from a general proposition, any one of its substitution instances). A more detailed discussion of the structure and content of logic would, however, lead too far afield in the context of this article.

6. Peano's axiom system as a basis for mathematics. Let us now consider a postulate system from which the entire arithmetic of the natural numbers can be derived. This system was devised by the Italian mathematician and logician G. Peano (1858–1932). The primitives of this system are the terms "0," "number," and "successor." While, of course, no definition of these terms is given within the theory, the symbol "0" is intended to designate the number 0 in its usual meaning, while the term "number" is meant to refer to the natural numbers 0, 1, 2, 3 $\cdots$ exclusively. By the successor of a natural number n, which will sometimes briefly be called n', is meant the natural number immediately following n in the natural order. Peano's system contains the following 5 postulates:

P1. 0 is a number
P2. The successor of any number is a number
P3. No two numbers have the same successor
P4. 0 is not the successor of any number
P5. If P is a property such that (a) 0 has the property P, and (b) whenever a number n has the property P, then the successor of n also has the property P, then every number has the property P.

The last postulate embodies the principle of mathematical induction and illustrates in a very obvious manner the enforcement of a mathematical "truth" by stipulation. The construction of elementary arithmetic on this basis begins with the definition of the various natural numbers. 1 is defined as the successor of 0, or briefly as $0'$; 2 as $1'$, 3 as $2'$, and so on. By virtue of P2, this process can be continued indefinitely; because of P3 (in combination with P5), it never leads back to one of the numbers previously defined, and in view of P4, it does not lead back to 0 either.

As the next step, we can set up a definition of addition which expresses in a precise form the idea that the addition of any natural number to some given number may be considered as a repeated addition of 1; the latter operation is readily expressible by means of the successor relation. This definition of addition runs as follows:

D1. (a) $n+0=n$; (b) $n+k'=(n+k)'$.

The two stipulations of this recursive definition completely determine the sum of any two integers. Consider, for example, the sum $3+2$. According to the definitions of the numbers 2 and 1, we have $3+2=3+1'=3+(0')'$; by D1 (b), $3+(0')'=(3+0')'=((3+0)')'$; but by D1 (a), and by the definitions of the numbers 4 and 5, $((3+0)')'=(3')'=4'=5$. This proof also renders more explicit and precise the comments made earlier in this paper on the truth of the proposition that $3+2=5$: Within the Peano system of arithmetic, its truth flows not merely from the definition of the concepts involved, but also from the postulates that govern these various concepts. (In our specific example, the postulates P1 and P2 are presupposed to guarantee that 1, 2, 3, 4, 5 are numbers in Peano's system; the general proof that D1 determines the sum of any two numbers also makes use of P5.) If we call the postulates and definitions of an axiomatized theory the "stipulations" concerning the concepts of that theory, then we may say now that the propositions of the arithmetic of the natural numbers are true by virtue of the stipulations which have been laid down initially for the arithmetical concepts. (Note, incidentally, that our proof of the formula "$3+2=5$" repeatedly made use of the transitivity of identity; the latter is accepted here as one of the rules of logic which may be used in the proof of any arithmetical theorem; it is, therefore, included among Peano's postulates no more than any other principle of logic.)

Now, the multiplication of natural numbers may be defined by means of the following recursive definition, which expresses in a rigorous form the idea that a product nk of two integers may be considered as the sum of k terms each of which equals n.

D2. (a) $n\cdot 0=0$; (b) $n\cdot k'=n\cdot k+n$.

It now is possible to prove the familiar general laws governing addition and multiplication, such as the commutative, associative, and distributive laws $(n+k=k+n,\ n\cdot k=k\cdot n;\ n+(k+l)=(n+k)+l,\ n\cdot(k\cdot l)=(n\cdot k)\cdot l;\ n\cdot(k+l)=(n\cdot k)+(n\cdot l))$.—In terms of addition and multiplication, the inverse operations of subtraction and division can then be defined. But it turns out that these "cannot always be performed"; *i.e.*, in contradistinction to the sum and the product, the difference and the quotient are not defined for every couple of numbers; for example, $7-10$ and $7\div 10$ are undefined. This situation suggests an enlargement of the number system by the introduction of negative and of rational numbers.

It is sometimes held that in order to effect this enlargement, we have to "assume" or else to "postulate" the existence of the desired additional kinds of numbers with properties that make them fit to fill the gaps of subtraction and division. This method of simply postulating what we want has its advantages; but, as Bertrand Russell [5] puts it, they are the same as the advantages of theft over honest toil; and it is a remarkable fact that the negative as well as the ra-

tional numbers can be obtained from Peano's primitives by the honest toil of constructing explicit definitions for them, without the introduction of any new postulates or assumptions whatsoever. Every positive and negative integer—in contradistinction to a natural number which has no sign—is definable as a certain set of ordered couples of natural numbers; thus, the integer $+2$ is definable as the set of all ordered couples (m, n) of natural numbers where $m=n+2$; the integer -2 is the set of all ordered couples (m, n) of natural numbers with $n=m+2$.—Similarly, rational numbers are defined as classes of ordered couples of integers.—The various arithmetical operations can then be defined with reference to these new types of numbers, and the validity of all the arithmetical laws governing these operations can be proved by virtue of nothing more than Peano's postulates and the definitions of the various arithmetical concepts involved.

The much broader system thus obtained is still incomplete in the sense that not every number in it has a square root, and more generally, not every algebraic equation whose coefficients are all numbers of the system has a solution in the system. This suggests further expansions of the number system by the introduction of real and finally of complex numbers. Again, this enormous extension can be effected by mere definition, without the introduction of a single new postulate [6]. On the basis thus obtained, the various arithmetical and algebraic operations can be defined for the numbers of the new system, the concepts of function, of limit, of derivative and integral can be introduced, and the familiar theorems pertaining to these concepts can be proved, so that finally the huge system of mathematics as here delimited rests on the narrow basis of Peano's system: Every concept of mathematics can be defined by means of Peano's three primitives, and every proposition of mathematics can be deduced from the five postulates enriched by the definitions of the non-primitive terms [6*a*]. These deductions can be carried out, in most cases, by means of nothing more than the principles of formal logic; the proof of some theorems concerning real numbers, however, requires one assumption which is not usually included among the latter. This is the so-called axiom of choice. It asserts that given a class of mutually exclusive classes, none of which is empty, there exists at least one class which has exactly one element in common with each of the given classes. By virtue of this principle and the rules of formal logic, the content of all of mathematics can thus be derived from Peano's modest system—a remarkable achievement in systematizing the content of mathematics and clarifying the foundations of its validity.

7. Interpretations of Peano's primitives. As a consequence of this result, the whole system of mathematics might be said to be true by virtue of mere definitions (namely, of the non-primitive mathematical terms) provided that the five Peano postulates are true. However, strictly speaking, we cannot, at this juncture, refer to the Peano postulates as propositions which are either true or false, for they contain three primitive terms which have not been assigned any specific meaning. All we can assert so far is that any specific interpretation of the primi-

tives which satisfies the five postulates—*i.e.*, turns them into true statements—will also satisfy all the theorems deduced from them. But for Peano's system, there are several—indeed, infinitely many—interpretations which will do this. For example, let us understand by 0 the origin of a half-line, by the successor of a point on that half-line the point 1 cm. behind it, counting from the origin, and by a number any point which is either the origin or can be reached from it by a finite succession of steps each of which leads from one point to its successor. It can then readily be seen that all the Peano postulates as well as the ensuing theorems turn into true propositions, although the interpretation given to the primitives is certainly not the customary one, which was mentioned earlier. More generally, it can be shown that every progression of elements of any kind provides a true interpretation, or a "model," of the Peano system. This example illustrates our earlier observation that a postulate system cannot be regarded as a set of "implicit definitions" for the primitive terms: The Peano system permits of many different interpretations, whereas in everyday as well as in scientific language, we attach one specific meaning to the concepts of arithmetic. Thus, *e.g.*, in scientific and in everyday discourse, the concept 2 is understood in such a way that from the statement "Mr. Brown as well as Mr. Cope, but no one else is in the office, and Mr. Brown is not the same person as Mr. Cope," the conclusion "Exactly two persons are in the office" may be validly inferred. But the stipulations laid down in Peano's system for the natural numbers, and for the number 2 in particular, do not enable us to draw this conclusion; they do not "implicitly determine" the customary meaning of the concept 2 or of the other arithmetical concepts. And the mathematician cannot acquiesce at this deficiency by arguing that he is not concerned with the customary meaning of the mathematical concepts; for in proving, say, that every positive real number has exactly two real square roots, he is himself using the concept 2 in its customary meaning, and his very theorem cannot be proved unless we presuppose more about the number 2 than is stipulated in the Peano system.

If therefore mathematics is to be a correct theory of the mathematical concepts in their intended meaning, it is not sufficient for its validation to have shown that the entire system is derivable from the Peano postulates plus suitable definitions; rather, we have to inquire further whether the Peano postulates are actually true when the primitives are understood in their customary meaning. This question, of course, can be answered only after the customary meaning of the terms "0," "natural number," and "successor" has been clearly defined. To this task we now turn.

8. Definition of the customary meaning of the concepts of arithmetic in purely logical terms. At first blush, it might seem a hopeless undertaking to try to define these basic arithmetical concepts without presupposing other terms of arithmetic, which would involve us in a circular procedure. However, quite rigorous definitions of the desired kind can indeed be formulated, and it can be shown that for the concepts so defined, all Peano postulates turn into true state-

ments. This important result is due to the research of the German logician G. Frege (1848–1925) and to the subsequent systematic and detailed work of the contemporary English logicians and philosophers B. Russell and A. N. Whitehead. Let us consider briefly the basic ideas underlying these definitions [7].

A natural number—or, in Peano's term, a number—in its customary meaning can be considered as a characteristic of certain *classes* of objects. Thus, *e.g.*, the class of the apostles has the number 12, the class of the Dionne quintuplets the number 5, any couple the number 2, and so on. Let us now express precisely the meaning of the assertion that a certain class C has the number 2, or briefly, that $n(C)=2$. Brief reflection will show that the following definiens is adequate in the sense of the customary meaning of the concept 2: There is some object x and some object y such that (1) $x \epsilon C$ (*i.e.*, x is an element of C) and $y \epsilon C$, (2) $x \neq y$, and (3) if z is any object such that $z \epsilon C$, then either $z=x$ or $z=y$. (Note that on the basis of this definition it becomes indeed possible to infer the statement "The number of persons in the office is 2" from "Mr. Brown as well as Mr. Cope, but no one else is in the office, and Mr. Brown is not identical with Mr. Cope"; C is here the class of persons in the office.) Analogously, the meaning of the statement that $n(C)=1$ can be defined thus: There is some x such that $x \epsilon C$, and any object y such that $y \epsilon C$, is identical with x. Similarly, the customary meaning of the statement that $n(C)=0$ is this: There is no object such that $x \epsilon C$.

The general pattern of these definitions clearly lends itself to the definition of any natural number. Let us note especially that in the definitions thus obtained, the definiens never contains any arithmetical term, but merely expressions taken from the field of formal logic, including the signs of identity and difference. So far, we have defined only the meaning of such phrases as "$n(C)=2$," but we have given no definition for the numbers 0, 1, 2, $\cdots$ apart from this context. This desideratum can be met on the basis of the consideration that 2 is that property which is common to all couples, *i.e.*, to all classes C such that $n(C)=2$. This common property may be conceptually represented by the class of all those classes which share this property. Thus we arrive at the definition: 2 is the class of all couples, *i.e.*, the class of all classes C for which $n(C)=2$.—This definition is by no means circular because the concept of couple—in other words, the meaning of "$n(C)=2$"—has been previously defined without any reference to the number 2. Analogously, 1 is the class of all unit classes, *i.e.*, the class of all classes C for which $n(C)=1$. Finally, 0 is the class of all null classes, *i.e.*, the class of all classes without elements. And as there is only one such class, 0 is simply the class whose only element is the null class. Clearly, the customary meaning of any given natural number can be defined in this fashion [8]. In order to characterize the intended interpretation of Peano's primitives, we actually need, of all the definitions here referred to, only that of the number 0. It remains to define the terms "successor" and "integer."

The definition of "successor," whose precise formulation involves too many

niceties to be stated here, is a careful expression of a simple idea which is illustrated by the following example: Consider the number 5, *i.e.*, the class of all quintuplets. Let us select an arbitrary one of these quintuplets and add to it an object which is not yet one of its members. 5′, the successor of 5, may then be defined as the number applying to the set thus obtained (which, of course, is a sextuplet). Finally, it is possible to formulate a definition of the customary meaning of the concept of natural number; this definition, which again cannot be given here, expresses, in a rigorous form, the idea that the class of the natural numbers consists of the number 0, its successor, the successor of that successor, and so on.

If the definitions here characterized are carefully written out—this is one of the cases where the techniques of symbolic, or mathematical, logic prove indispensable—it is seen that the definiens of every one of them contains exclusively terms from the field of pure logic. In fact, it is possible to state the customary interpretation of Peano's primitives, and thus also the meaning of every concept definable by means of them—and that includes every concept of mathematics—in terms of the following 7 expressions, in addition to variables such as "x" and "C": *not, and, if—then; for every object* x it is the case that . . . ; *there is some object* x such that . . . ; x is an *element* of class C; *the class of all things* x *such that* . . . . And it is even possible to reduce the number of logical concepts needed to a mere four: The first three of the concepts just mentioned are all definable in terms of "*neither—nor*," and the fifth is definable by means of the fourth and "*neither—nor*." Thus, all the concepts of mathematics prove definable in terms of four concepts of pure logic. (The definition of one of the more complex concepts of mathematics in terms of the four primitives just mentioned may well fill hundreds or even thousands of pages; but clearly this affects in no way the theoretical importance of the result just obtained; it does, however, show the great convenience and indeed practical indispensability for mathematics of having a large system of highly complex defined concepts available.)

9. The truth of Peano's postulates in their customary interpretation. The definitions characterized in the preceding section may be said to render precise and explicit the customary meaning of the concepts of arithmetic. Moreover—and this is crucial for the question of the validity of mathematics—it can be shown that the Peano postulates all turn into true propositions if the primitives are construed in accordance with the definitions just considered.

Thus, P1 (0 is a number) is true because the class of all numbers—*i.e.*, natural numbers—was defined as consisting of 0 and all its successors. The truth of P2 (The successor of any number is a number) follows from the same definition. This is true also of P5, the principle of mathematical induction. To prove this, however, we would have to resort to the precise definition of "integer" rather than the loose description given of that definition above. P4 (0 is not the successor of any number) is seen to be true as follows: By virtue of the definition of "successor," a number which is a successor of some number can apply only to

classes which contain at least one element; but the number 0, by definition, applies to a class if and only if that class is empty.—While the truth of P1, P2, P4, P5 can be inferred from the above definitions simply by means of the principles of logic, the proof of P3 (No two numbers have the same successor) presents a certain difficulty. As was mentioned in the preceding section, the definition of the successor of a number n is based on the process of adding, to a class of n elements, one element not yet contained in that class. Now if there should exist only a finite number of things altogether then this process could not be continued indefinitely, and P3, which (in conjunction with P1 and P2) implies that the integers form an infinite set, would be false. Russell's way of meeting this difficulty [9] was to introduce a special "axiom of infinity," which stipulates, in effect, the existence of infinitely many objects and thus makes P3 demonstrable. The axiom of infinity can be formulated in purely logical terms and may therefore be considered as a postulate of logic; however, it certainly does not belong to the generally recognized principles of logic; and it thus introduces a foreign element into the otherwise unexceptionable derivation of the Peano postulates from pure logic. Recently, however, it has been shown [10] that a suitable system of logical principles can be set up which is even less comprehensive than the rules of logic which are commonly used [11], and in which the existence of infinitely many objects can be proved without the need for a special axiom.

10. Mathematics as a branch of logic. As was pointed out earlier, all the theorems of arithmetic, algebra, and analysis can be deduced from the Peano postulates and the definitions of those mathematical terms which are not primitives in Peano's system. This deduction requires only the principles of logic plus, in certain cases, the axiom of choice. By combining this result with what has just been said about the Peano system, the following conclusion is obtained, which is also known as *the thesis of logicism concerning the nature of mathematics*:

Mathematics is a branch of logic. It can be derived from logic in the following sense:

a. All the concepts of mathematics, *i.e.* of arithmetic, algebra, and analysis, can be defined in terms of four concepts of pure logic.

b. All the theorems of mathematics can be deduced from those definitions by means of the principles of logic (including the axiom of choice).

In this sense it can be said that the propositions of the system of mathematics as here delimited are true by virtue of the definitions of the mathematical concepts involved, or that they make explicit certain characteristics with which we have endowed our mathematical concepts by definition. The propositions of mathematics have, therefore, the same unquestionable certainty which is typical of such propositions as "All bachelors are unmarried," but they also share the complete lack of empirical content which is associated with that certainty: The propositions of mathematics are devoid of all factual content; they convey no information whatever on any empirical subject matter.

11. On the applicability of mathematics to empirical subject matter. This result seems to be irreconcilable with the fact that after all mathematics has proved to be eminently applicable to empirical subject matter, and that indeed the greater part of present-day scientific knowledge has been reached only through continual reliance on and application of the propositions of mathematics.—Let us try to clarify this apparent paradox by reference to some examples.

Suppose that we are examining a certain amount of some gas, whose volume v, at a certain fixed temperature, is found to be 9 cubic feet when the pressure p is 4 atmospheres. And let us assume further that the volume of the gas for the same temperature and $p=6$ *at.*, is predicted by means of Boyle's law. Using elementary arithmetic we reason thus: For corresponding values of v and p, $vp=c$, and $v=9$ when $p=4$; hence $c=36$: Therefore, when $p=6$, then $v=6$. Suppose that this prediction is borne out by subsequent test. Does that show that the arithmetic used has a predictive power of its own, that its propositions have factual implications? Certainly not. All the predictive power here deployed, all the empirical content exhibited stems from the initial data and from Boyle's law, which asserts that $vp=c$ for *any* two corresponding values of v and p, hence also for $v=9$, $p=4$, and for $p=6$ and the corresponding value of v [12]. The function of the mathematics here applied is not predictive at all; rather, it is analytic or explicative: it renders explicit certain assumptions or assertions which are included in the content of the premises of the argument (in our case, these consist of Boyle's law plus the additional data); mathematical reasoning reveals that those premises contain—hidden in them, as it were,—an assertion about the case as yet unobserved. In accepting our premises—so arithmetic reveals—we have—knowingly or unknowingly—already accepted the implication that the p-value in question is 6. Mathematical as well as logical reasoning is a conceptual technique of making explicit what is implicitly contained in a set of premises. The conclusions to which this technique leads assert nothing that is *theoretically new* in the sense of not being contained in the content of the premises. But the results obtained may well be *psychologically new*: we may not have been aware, before using the techniques of logic and mathematics, what we committed ourselves to in accepting a certain set of assumptions or assertions.

A similar analysis is possible in all other cases of applied mathematics, including those involving, say, the calculus. Consider, for example, the hypothesis that a certain object, moving in a specified electric field, will undergo a constant acceleration of 5 feet/sec^2. For the purpose of testing this hypothesis, we might derive from it, by means of two successive integrations, the prediction that if the object is at rest at the beginning of the motion, then the distance covered by it at any time t is $\frac{5}{2}t^2$ feet. This conclusion may clearly be psychologically new to a person not acquainted with the subject, but it is not theoretically new; the content of the conclusion is already contained in that of the hypothesis about the constant acceleration. And indeed, here as well as in the case of the compression of a gas, a failure of the prediction to come true would be considered as indica-

tive of the factual incorrectness of at least one of the premises involved (*f.ex.*, of Boyle's law in its application to the particular gas), but never as a sign that the logical and mathematical principles involved might be unsound.

Thus, in the establishment of empirical knowledge, mathematics (as well as logic) has, so to speak, the function of a theoretical juice extractor: the techniques of mathematical and logical theory can produce no more juice of factual information than is contained in the assumptions to which they are applied; but they may produce a great deal more juice of this kind than might have been anticipated upon a first intuitive inspection of those assumptions which form the raw material for the extractor.

At this point, it may be well to consider briefly the status of those mathematical disciplines which are not outgrowths of arithmetic and thus of logic; these include in particular topology, geometry, and the various branches of abstract algebra, such as the theory of groups, lattices, fields, *etc.* Each of these disciplines can be developed as a purely deductive system on the basis of a suitable set of postulates. If P be the conjunction of the postulates for a given theory, then the proof of a proposition T of that theory consists in deducing T from P by means of the principles of formal logic. What is established by the proof is therefore not the truth of T, but rather the fact that T is true provided that the postulates are. But since both P and T contain certain primitive terms of the theory, to which no specific meaning is assigned, it is not strictly possible to speak of the truth of either P or T; it is therefore more adequate to state the point as follows: If a proposition T is logically deduced from P, then every specific interpretation of the primitives which turns all the postulates of P into true statements, will also render T a true statement.—Up to this point, the analysis is exactly analogous to that of arithmetic as based on Peano's set of postulates. In the case of arithmetic, however, it proved possible to go a step further, namely to define the customary meanings of the primitives in terms of purely logical concepts and to show that the postulates—and therefore also the theorems—of arithmetic are unconditionally true by virtue of these definitions. An analogous procedure is not applicable to those disciplines which are not outgrowths of arithmetic: The primitives of the various branches of abstract algebra have no specific "customary meaning"; and if geometry in its customary interpretation is thought of as a theory of the structure of physical space, then its primitives have to be construed as referring to certain types of physical entities, and the question of the truth of a geometrical theory in this interpretation turns into an *empirical* problem [13]. For the purpose of applying any one of these non-arithmetical disciplines to some specific field of mathematics or empirical science, it is therefore necessary first to assign to the primitives some specific meaning and then to ascertain whether in this interpretation the postulates turn into true statements. If this is the case, then we can be sure that all the theorems are true statements too, because they are logically derived from the postulates and thus simply explicate the content of the latter in the given interpretation.—In their application to empirical subject matter, therefore, these mathematical theories no less than

those which grow out of arithmetic and ultimately out of pure logic, have the function of an analytic tool, which brings to light the implications of a given set of assumptions but adds nothing to their content.

But while mathematics in no case contributes anything to the content of our knowledge of empirical matters, it is entirely indispensable as an instrument for the validation and even for the linguistic expression of such knowledge: The majority of the more far-reaching theories in empirical science—including those which lend themselves most eminently to prediction or to practical application—are stated with the help of mathematical concepts; the formulation of these theories makes use, in particular, of the number system, and of functional relationships among different metrical variables. Furthermore, the scientific test of these theories, the establishment of predictions by means of them, and finally their practical application, all require the deduction, from the general theory, of certain specific consequences; and such deduction would be entirely impossible without the techniques of mathematics which reveal what the given general theory implicitly asserts about a certain special case.

Thus, the analysis outlined on these pages exhibits the system of mathematics as a vast and ingenious conceptual structure without empirical content and yet an indispensable and powerful theoretical instrument for the scientific understanding and mastery of the world of our experience.

References

1. A discussion of the status of geometry is given in my article, Geometry and Empirical Science, American Mathematical Monthly, vol. 52, pp. 7–17, 1945.

2. The objection is sometimes raised that without certain types of experience, such as encountering several objects of the same kind, the integers and the arithmetical operations with them would never have been invented, and that therefore the propositions of arithmetic do have an empirical basis. This type of argument, however, involves a confusion of the logical and the psychological meaning of the term "basis." It may very well be the case that certain experiences occasion psychologically the formation of arithmetical ideas and in this sense form an empirical "basis" for them; but this point is entirely irrelevant for the logical questions as to the *grounds* on which the propositions of arithmetic may be accepted as true. The point made above is that no empirical "basis" or evidence whatever is needed to establish the truth of the propositions of arithmetic.

3. A precise account of the definition and the essential characteristics of the identity relation may be found in A. Tarski, Introduction to Logic, New York, 1941, Ch. III.

4. For a lucid and concise account of the axiomatic method, see A. Tarski, *l.c.*, Ch. VI.

5. Bertrand Russell, Introduction to Mathematical Philosophy, New York and London, 1919, p. 71.

6. For a more detailed account of the construction of the number system on Peano's basis, *cf.* Bertrand Russell, *l.c.*, esp. Chs. I and VII.—A rigorous and concise presentation of that construction, beginning, however, with the set of all integers rather than that of the natural numbers, may be found in G. Birkhoff and S. MacLane, A Survey of Modern Algebra, New York 1941, Chs. I, II, III, V.—For a general survey of the construction of the number system, *cf.* also J. W. Young, Lectures on the Fundamental Concepts of Algebra and Geometry, New York, 1911, esp. lectures X, XI, XII.

6*a*. As a result of very deep-reaching investigations carried out by K. Gödel it is known that arithmetic, and *a fortiori* mathematics, is an incomplete theory in the following sense: While all those propositions which belong to the classical systems of arithmetic, algebra, and analysis can

indeed be derived, in the sense characterized above, from the Peano postulates, there exist nevertheless other propositions which can be expressed in purely arithmetical terms, and which are true, but which cannot be derived from the Peano system. And more generally: For any postulate system of arithmetic (or of mathematics for that matter) which is not self-contradictory, there exist propositions which are true, and which can be stated in purely arithmetical terms, but which cannot be derived from that postulate system. In other words, it is impossible to construct a postulate system which is not self-contradictory, and which contains among its consequences all true propositions which can be formulated within the language of arithmetic.

This fact does not, however, affect the result outlined above, namely, that it is possible to deduce, from the Peano postulates and the additional definitions of non-primitive terms, all those propositions which constitute the classical theory of arithmetic, algebra, and analysis; and it is to these propositions that I refer above and subsequently as the propositions of mathematics.

7. For a more detailed discussion, *cf.* Russell, *l.c.*, Chs. II, III, IV. A complete technical development of the idea can be found in the great standard work in mathematical logic, A. N. Whitehead and B. Russell, Principia Mathematica, Cambridge, England, 1910–1913.—For a very precise recent development of the theory, see W. V. O. Quine, Mathematical Logic, New York 1940.—A specific discussion of the Peano system and its interpretations from the viewpoint of semantics is included in R. Carnap, Foundations of Logic and Mathematics, International Encyclopedia of Unified Science, vol. I, no. 3, Chicago, 1939; especially sections 14, 17, 18.

8. The assertion that the definitions given above state the "customary" meaning of the arithmetical terms involved is to be understood in the logical, not the psychological sense of the term "meaning." It would obviously be absurd to claim that the above definitions express "what everybody has in mind" when talking about numbers and the various operations that can be performed with them. What is achieved by those definitions is rather a "logical reconstruction" of the concepts of arithmetic in the sense that if the definitions are accepted, then those statements in science and everyday discourse which involve arithmetical terms can be interpreted coherently and systematically in such a manner that they are capable of objective validation. The statement about the two persons in the office provides a very elementary illustration of what is meant here.

9. *Cf.* Bertrand Russell, *l.c.*, p. 24 and Ch. XIII.

10. This result has been obtained by W. V. O. Quine; *cf.* his Mathematical Logic, New York, 1940.

11. The principles of logic developed in Quine's work and in similar modern systems of formal logic embody certain restrictions as compared with those logical rules which had been rather generally accepted as sound until about the turn of the 20th century. At that time, the discovery of the famous paradoxes of logic, especially of Russell's paradox (*cf.* Russell, *l.c.*, Ch. XIII) revealed the fact that the logical principles implicit in customary mathematical reasoning involved contradictions and therefore had to be curtailed in one manner or another.

12. Note that we may say "hence" by virtue of the rule of substitution, which is one of the rules of logical inference.

13. For a more detailed discussion of this point, *cf.* the article mentioned in reference 1.

TWO DIMENSIONS

J. L. COOLIDGE, Harvard University

1. The difficulty. A number of years ago I heard a distinguished geometer deliver the dictum, "In geometry the only functions in which we are interested are analytic functions, and the only interesting theorems are those which are true in any number of dimensions." I think that the two parts of this statement can be treated separately, as I cannot see any connection between them.

I heartily agree with the first part. The geometer does not like to be bothered with questions of analyticity; let the analysts worry about them. Some differential geometers, in an effort to appear liberal, change the formula and say, "We assume the existence of all derivatives which appear." This would seem fairly obvious to most persons. In any case, if the geometer has three or four good derivatives he does not much care whether there is a discontinuity in the fifth.

The case seems to me very different with regard to the second part of the statement: "No theorem is interesting unless it hold in any number of dimensions." Of course what is involved here is a definition of "interesting." A mathematician hesitates to say what he means by "interesting" and he carefully refrains from asking a non-mathematician what *he* means by "interesting" in mathematics, as he knows only too well what the answer would be. If one grants that part of the definition of "interesting" is that the statement holds in any number of dimensions, there is nothing more to be said. I am interested in the Pythagorean theorem and the four color problem; my opponent is not. Let us at least part as friends. To me it is a necessary condition for something to be interesting that it should be specific enough to present a definite and clear-cut picture to the mind, and that it should be general enough to include a fairly large variety of material.

What are the sufficient conditions? I don't know. But I will not accept application to any number of dimensions as necessary, but will maintain that geometry in two dimensions is not only simpler than geometry in a higher space, but frequently richer and more significant. Let me try to justify this assumption.

2. Examples. Let us begin with projective geometry. We have a set of objects in one to one correspondence with the values of the ratios $n+1$ homogeneous coordinates, not all zero. These we call "points of an n-space" or S_n. Points whose coordinates are linearly dependent on those of two given points form a line, those dependent on three linearly independent ones a plane. If there be $k+1$ linearly independent points we have a k-spread or S_k. The coordinates of the points of a k-spread are connected by $n-k$ homogeneous linear equations. The points whose coordinates are connected by a single quadratic equation, say $a_{ij}x^ix^j=0$ are said to form a *hyperquadric*. Hyperquadrics are classified under the projective group by the rank of the matrix $\|a_{ij}\|$.

So far all serene. There is no virtue in any one number of dimensions, but · · · · In two dimensions we have the theorems of Pascal and Brianchon

for hexagons inscribed or circumscribed to conics. There is nothing really comparable in higher space, and for that reason the study of the conics is richer than that of the other hyperquadrics, even if we raise our eyebrows at the usual statement that Pascal discovered more than four hundred corollaries to his theorem.

At this point an opponent can come back at me rather hard with the remark that Desargues' two-triangle theorem can be derived for planes contained in a three-space from the simplest assumptions about points, lines and planes in the space, whereas it cannot be derived from assumptions about points and lines in the plane itself. This is perfectly true. I am willing to concede that a plane that can be contained in a three-space is richer than one that cannot, but I cannot see that the main thesis is affected. I think also that Desargues' figure of two triangles is more generally important that Stephanos' configuration of three desmic tetrahedra, the nearest three-dimensional analogue.

Let us next look at algebraic curves. The most teasing thing about these is their singular points. These can be of various sorts and the earlier English geometers invented a variety of names: acnodes, crunodes, ceratoid cusps and the like. But Noether showed that any singular point of a plane algebraic curve could be looked upon as a union of infinitely near ordinary singular points, or as the limit of such points as they approached to final positions, a singular point being ordinary when the tangents are distinct. The numbers here involved are not at all arbitrary but appear in perfectly definite fashion in the power series development of the curve in the vicinity of the singularity. This concept draws the whole theory together and enables us to handle it with confidence.

I am not aware that there is anything at all corresponding to this in the study of singularities of other algebraic varieties. Step up but one dimension. An algebraic surface may have isolated singular points, or singular curves, or singular points of singular curves or pinch points on curves. I don't think the possibilities have all been classified or sub-divided. In the birational theory we transform to surfaces with nothing worse than double curves with triple points on them, like the corners of a room, but in the projective theory there seems no simple guiding principle.

A Cremona transformation of a plane or space is one that is algebraic and rational, with a rational algebraic inverse. Now Noether has shown that in the plane any Cremona transformation can be factored into linear and quadratic transformations, but in 1912 Hilda Hudson proved that there was no corresponding reduction of Cremona transformations in space. In a sense this makes the space theory richer, and the number of studies of it is much larger, but Noether's theorem gives an invaluable central motif.

Let us compare a circle and a sphere. Each is the locus of points at a given distance from a fixed point of the space in question. In each case the tangent variety is perpendicular to the radius, the polar plane is perpendicular to the line joining the pole with the centre. The fundamental relation of two circles or two spheres is orthogonal intersection; linear systems are composed of circles or spheres cutting others orthogonally. The theory of inversion is the same in

both cases and the general circle or sphere transformation can be factored into a number of inversions and reflections in planes. The identity seems complete. Not quite. All angles inscribed in the same circular arc are equal; the sphere has no corresponding property. The results of this are incalculable. The geometry of the tetrahedron is meager indeed compared with the fantastically rich geometry of the triangle. The constancy of the inscribed angle is much the most useful property of the circle. It is curious to observe that it depends on the euclidean parallel postulate; had our customary geometry been Lobachevskian or Riemannian we should have been without it.

Let us finally look at conformal transformations. A transformation is conformal if it is a point to point transformation which keeps angles invariant. A conformal transformation of space carries a point to a point and keeps invariant the angle of intersection of two surfaces. It will therefore carry two surfaces which intersect at right angles into two other such surfaces, and so carry a triply orthogonal surface system into another such. But Dupin's theorem tells us that in a triply orthogonal system the various surfaces meet in their lines of curvature. Hence a conformal transformation will carry lines of curvature into lines of curvature, and so carry a surface, all of whose curves are lines of curvature, into another such surface. But the only surfaces with this property are spheres and planes. This gives us the theorem due, I think, to Liouville, that the only conformal transformations of three-space are sphere transformations.

Now look at two dimensions. Here it is well to use minimal coordinates

$$z = x + iy \qquad \bar{z} = x - iy.$$

A conformal transformation of the plane must carry a self-perpendicular direction into a self-perpendicular direction. The differential equation for such is

$$dzd\bar{z} = 0$$

so this must be invariant. Let our transformation be

$$z' = z'(z, \bar{z}) \qquad \bar{z}' = \bar{z}'(z, \bar{z})$$

$$dz'd\bar{z}' = F(z, \bar{z})dzd\bar{z}$$

$$\frac{\partial z'}{\partial z}\frac{\partial \bar{z}'}{\partial z} = 0; \qquad \frac{\partial z'}{\partial \bar{z}}\frac{\partial \bar{z}'}{\partial \bar{z}} = 0.$$

Then either

$$z' = z'(z); \qquad \bar{z}' = \bar{z}'(\bar{z}')$$

or

$$z' = z'(\bar{z}) \qquad \bar{z}' = \bar{z}'(\bar{z}).$$

Here we have the most striking case of the divergence of plane geometry from the geometry of higher space. The general conformal transformation of three-space depends on ten parameters; the most general conformal transformation of the plane depends on an infinite number of parameters, that is, on an arbitrary

function of the complex variable. It is worth mentioning in this connection that there is a prodigious amount of literature of conformal transformations of the plane, generally called by those most interested Conforme Abbildung.

I will also mention in passing a curious and little known theorem of Clifford's which comes from the use of these same minimal parameters, and which, of course, has no analogue in higher dimensions. Suppose that we have n lines through each of the circular points at infinity (not including the infinite line itself). They will have n^2 intersections. A set of n points, each of which lies on just one line of each set shall be called a "group" of intersections. There will be $n!$ such groups. Then the product of the distances of *any* point in the plane from the intersections of a group is independent of the group chosen. For instance, the product of the distances of any point in the plane from the real foci of an ellipse is equal to the product of its distances from the two imaginary foci.

3. Underlying reasons. It would not be difficult to find other examples of this sort of thing. It is more significant to try to see if there be any inner reason for the phenomenon. I can only grope.

Let me give two different proofs of two of the theorems. First, Liouville's for conformal transformations of three-space. In such a transformation, a self-perpendicular direction will go into a self-perpendicular direction. Hence on any surface the minimal curves, those whose tangents are self-perpendicular, will go into the corresponding curves on another surface. Hence a surface which has but a single set of such curves will go into another surface of this sort. But the only surfaces with a single set of minimal curves is a minimal developable, one whose generators are minimal straight lines. Hence a minimal developable will go into a minimal developable. It follows then that a set of minimal developables tangent at one point will go into a similar set. But such a set of developables have a common minimal line. Hence, finally, a conformal transformation of three-space will carry a minimal line into a minimal line.

In the plane there is only a one-parameter set of minimal lines to be carried, but a conformal transformation of space must keep invariant a whole three-parameter family of these lines, and this great invariance hampers the freedom of the system seriously.

Secondly, let us look at the equality of all angles in a circular arc. Suppose that in three-space we have five points which lie on a sphere $P_1P_2 \cdots P_5$. Let the distance of P_iP_j be d_{ij}. Then it is well known that

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 \\ d_{21}^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{51}^2 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0^2 \end{vmatrix} = 0.$$

This equation becomes rather complicated when multiplied out. There is a similar equation for any number of dimensions. But in the single case of two dimensions, when it is a question of four points lying on a circle, the determinant is factorable, and we have

$$(d_{12}d_{34} + d_{13}d_{24} + d_{14}d_{23})(-d_{12}d_{34} + d_{13}d_{24} + d_{14}d_{23})(d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23})$$
$$(d_{12}d_{34} + d_{13}d_{24} - d_{14}d_{23}) = 0.$$

If we set the appropriate one of these factors equal to zero, we have Ptolemy's theorem that the necessary and sufficient condition that the vertices of a quadrilateral lie on a circle is that the sum of the products of the opposite sides shall be equal to the product of the diagonals, from which the equality of the angles is easily deduced.

A correspondent kindly calls to my attention the following example which is simpler, if perhaps less instructive. If we have a triangle the lengths of whose sides are a, b, c, the square of the area, when expressed in determinant form is

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix}.$$

When this is expanded, it can be factored into the factors which appear in Hero's classical formula. The expression for the volume of a tetrahedron in terms of the lengths of its edges is unfactorable.

I conclude from these cases that when the data are comparatively simple a greater number of techniques are available, and so a greater variety of interesting results may be found. Or when there are too many dimensions or too many parameters involved, the great generality excludes the use of interesting methods which are available in simpler cases. I am not sure that this by any means covers the matter, but it seems to me a first suggestion.

I think it is also true that this same phenomenon appears in other parts of mathematics. In analysis the theory of functions of a single complex variable is a peculiarly compact and well integrated subject. The two corner stones here appear to be Cauchy's integral, and the Riemann surfaces, both of which involve the topology of the plane. Nothing so helpful is available when there are more variables. In the theory of differential equations we need not consider equations of higher than the second order to have most of the important material, nor does this seem to spring from the fact that we need only one differentiation for velocities and two for accelerations.

In algebra determinants, which are essentially two dimensional arrays, are of incalculable use. Various mathematicians have occupied themselves with three or more dimensional determinants; I could never learn that such things were of any real importance. The underlying fact here seems to be that when we are occupied with a certain number of functions of a certain number of variables

the question of their independence is vital, and this depends on the rank of a matrix, that is to say, the vanishing or non-vanishing of determinants.

The conclusion of the matter seems rather bleak. Modern mathematics is extremely abstract and extremely general. That is excellent. We always wish to know the extreme limits to which any truth will stretch. But I personally have the depressing feeling again and again, "Well, where do we go from here?" and I don't find the answer. Historically, progress is always from the particular to the general. I do not know of a single interesting theorem dealing with the geometry of the plane which was first found as a special case of something that holds in higher space, and there are plenty of interesting plane theorems that do not step up. With Fermat's and Goldbach's theorems still mocking us and the four color problem still unsolved there still remain some simple puzzles to tax our ingenuity.

AN ELEMENTARY CONSTRUCTIVE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

P. C. ROSENBLOOM, Brown University

1. Introduction. In this paper we shall give a method of constructing from a given polynomial $P(z)$ by rational operations a sequence of complex numbers $\{z_n\}$ which satisfies the Cauchy convergence criterion and such that $P(z_n)$ converges to zero. All of this part of the proof is constructive and completely elementary. The existence of a root of $P(z)$ then follows immediately from the definition of the real number system by means of Cantor's regular sequences. (See for example [4].) Brouwer and deLoor [1] gave an "intuitionistic" proof (see also [2]), but made use of the properties of continuous curves and functions, while Weyl's proof [5] makes use of integration, and in particular, the formula

$$\frac{1}{2\pi i}\int_C \frac{f'(z)dz}{f(z)} = n$$

for the number of zeros of the polynomial $f(z)$ enclosed by the contour C. If these preliminaries were developed in full, their proofs would be much more complicated than they appear to be. In our proof the only appearance of limiting processes is in the very definition of the real number system.

Our main device is the use of summing over the lattice points of squares instead of integration. This idea can be used to replace other function-theoretic proofs by elementary algebraic proofs. We suspect that, for example, the Schottky-Landau theorem can be proved for polynomials in this way. Pedagogically, however, it would be better to preface our present proof by the following function-theoretic argument after which it is modeled. If the polynomial $P(z)$ of degree N had no zeros, then $P'(z)/P(z)$ would be regular everywhere, and therefore

$$\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz = 0$$

for the circle C: $|z| = R$. But

$$\left|\frac{1}{2\pi i}\int_C \frac{P'(z)dz}{P(z)} - \frac{1}{2\pi i}\int_C \frac{Ndz}{z}\right| = O(R^{-1})$$

as $R\to\infty$, and

$$\frac{1}{2\pi i}\int_C \frac{Ndz}{z} = N.$$

These three results are incompatible, and therefore $P(z)$ must have a root. The minor details of the computations involved in the following argument will be more perspicuous if this skeleton key is kept in mind.

2. Sums over lattice points. If $f(z)$ is any function of the complex variable z defined on the square $C_a(z_0)$ with center at z_0 and sides equal to $2a$ and parallel to the coordinate axes, we define the nth sum of $f(z)$ over $C_a(z_0)$ as

$$S_n(f; z_0, a) = \frac{2a}{n}\sum_{k=1}^{n}\left[\{f(z_0 - a - ai + (2k-1)h) - f(z_0 + a + ai - (2k-1)h)\}\right.$$
$$\left. + i\{f(z_0 + a - ai + i(2k-1)h) - f(z_0 - a + ai - i(2k-1)h)\}\right],$$

where $h = a/n$. This is obtained by dividing each side of $C_a(z_0)$ into n equal segments, denoting by $z_1, \cdots, z_{4n+1} = z_1$ the points of subdivision taken in counterclockwise order and by $\zeta_k = (z_k + z_{k+1})/2$ the midpoint of the kth segment, and forming the sum

$$S_n(f; z_0, a) = \sum_{k=1}^{4n} f(\zeta_k)(z_{k+1} - z_k).$$

If we worked with circles instead of squares, we would have to introduce the exponential function and other foreign matter.

The following properties of this summing operation are almost trivial:

$$S_n(c; z_0, a) = 0 \qquad (c = \text{constant});$$
$$S_n(z; z_0, a) = 0;$$
$$S_n(1/z; 0, a) = \sum_{k=1}^{n} \frac{8ni}{n^2 + (n+1-2k)^2} = 2\pi_n i;$$
$$S_n(cf; z_0, a) = cS_n(f; z_0, a) \qquad (c = \text{constant});$$
$$S_n(f + g; z_0, a) = S_n(f; z_0, a) + S_n(g; z_0, a);$$
$$S_n(f; z_0, a) = \sum_{p,q=1}^{n} S_1\left(f; z_{pq}, \frac{a}{n}\right),$$

where the z_{pq}'s are the centers of the n^2 squares with sides equal to $2a/n$ obtained by dissecting $C_a(z_0)$ by lines parallel to the coordinate axes. We note that

$$\pi_n > \frac{n \cdot 4n}{n^2 + n^2} = 2.$$

Also, if $|f(z)| \leqq M$ at the points ζ_k, then

$$|S_n(f; z_0, a)| \leqq 8aM. \tag{1}$$

3. Cauchy's theorem. We now derive an analogue to Cauchy's theorem for rational functions. The details will be clearer if the classical proof of the latter, say as presented in Knopp [3], is borne in mind.

Let $f(z) = Q(z)/P(z)$, where $P(z) = \sum_{k=0}^{N} a_k z^k$, $Q(z) = \sum_{k=0}^{N} b_k z^k$; let $A = \max|a_k|$ and $B = \max|b_k|$ for $k = 0, 1, \cdots, N$. We shall maintain these notations throughout the rest of the paper.

LEMMA 1. *If* $|P(z)| \geqq m > 0$ *on the lattice points in and on the square* $C_a(0)$ *with the mesh* $h = a/n$, *and* $a < 2^{-1/2}$, *then*

$$|S_n(f; 0, a)| < \frac{64A^2Ba^3}{m^3(1 - \sqrt{2}a)}.$$

Proof. Let $\epsilon(z) = f(z) - f(0) - zf'(0)$. Then we have $S_n(f; 0, a) = S_n(\epsilon; 0, a)$. But

$$\epsilon(z) = \frac{a_0^2 Q(z) - a_0 b_0 P(z) - (a_0 b_1 - b_0 a_1) z P(z)}{a_0^2 P(z)}$$

$$= \frac{\sum_{k=2}^{N+1} (a_0^2 b_k - a_0 a_k b_0 - a_0 a_{k-1} b_1 + a_1 a_{k-1} b_0) z^k}{a_0^2 P(z)},$$

where $a_{N+1} = b_{N+1} = 0$. Therefore

$$|\epsilon(z)| \leqq \frac{4A^2B}{m^3} \sum_{k=2}^{N+1} (\sqrt{2}a)^k < \frac{8a^2BA^2}{m^3(1 - \sqrt{2}a)}$$

on the lattice points of $C_a(0)$. The lemma now follows from (1).

We must now eliminate the special position of the point 0 in this lemma.

LEMMA 2. *If* $|P(z)| \geqq m > 0$ *on the lattice points of* $C_a(z_0)$ *with the mesh* $h = a/n$, *and* $a < 2^{-1/2}$, $|z_0| \leqq R$, $R \geqq 1$, *then*

$$|S_n(f; z_0, a)| \leqq \frac{64\gamma^3 R^{3N} BA^2 a^3}{m^3(1 - \sqrt{2}a)},$$

where γ *is the binomial coefficient* $C_{N+1,q}$ *with* $q = [(N+1)/2]$.

Proof. Let $P_1(z) = P(z_0 + z) = \sum_{k=0}^{N} a_{k1} z^k$, $Q_1(z) = Q(z_0 + z) = \sum_{k=0}^{N} b_{k1} z^k$, $A_1 = \max|a_{k1}|$, and $B_1 = \max|b_{k1}|$. Then

$$(2) \qquad |a_{k1}| = \left|\frac{P^{(k)}(z_0)}{k!}\right| = \left|\sum_{n=0}^{N-k} a_{n+k}C_{n+k,k}z_0^n\right| \leqq AR^{N-k}\sum_{n=0}^{N-k} C_{n+k,k} \leqq \gamma AR^N.$$

Hence $A_1 \leqq \gamma AR^N$, and similarly $B_1 \leqq \gamma BR^N$. Then the above estimate follows by Lemma 1 applied to $f_1(z) = f(z_0+z)$.

THEOREM 1. *Let*

$$m = \min_{|z|,|i|\leqq n} \left|P\left(\frac{(k+il)a}{n}\right)\right|,$$

and let $\sqrt{2}a \leqq R$, $1 \leqq R$, $\sqrt{2}a < n$. *Then*

$$|S_n(f; 0, a)| \leqq \frac{64\gamma^3R^{3N}BA^2a^3}{m^3(n-\sqrt{2}a)}.$$

Proof. We have

$$|S_n(f; 0, a)| \leqq \sum_{p,q=1}^{n} \left|S_1\left(f; z_{pq}, \frac{a}{n}\right)\right| = n^2 \cdot \frac{64\gamma^3R^{3N}BA^2a^3}{m^3(1-(\sqrt{2}a)/n)n^3}.$$

This is the analogue of Cauchy's theorem, which is the limiting case as $n \to \infty$.

4. Estimates for m. We shall establish the following theorem.

THEOREM 2. *Let* $a_N = 1$ (*so that* $A \geqq 1$), $a \geqq 5NA$, $n > 2\sqrt{2}a$, *and let* m *be as in Theorem* 1. *Then*

$$m^3 < \frac{K}{n}, \quad \textit{where} \quad K = 2^{(3N/2)+6}\gamma^3A^3a^{3N+3}.$$

Proof. Let $f(z) = P'(z)/P(z)$, so that $Q(z) = P'(z)$, and $B = \max|ka_k| \leqq NA$. We can take $R = \sqrt{2}a$ in Theorem 1 and obtain $|S_n(f; 0, a)| < K_1/m^3n$, where $K_1 = 2^{(3N/2)+7}\gamma^3NA^3a^{3N+3}$. But

$$2\pi_n Ni = S_n(N/z; 0, a) = S_n(f; 0, a) + S_n((NP - zP')/zP; 0, a).$$

Now

$$|NP - zP'| = \left|\sum_{k=0}^{N-1}(N-k)a_kz^k\right| \leqq NA\sum_{k=0}^{N-1}|z|^k \leqq N^2A|z|^{N-1}$$

if $|z| \geqq 1$, and

$$|P(z)| \geqq |z|^N - \sum_{k=0}^{N-1}|a_k||z|^k \geqq |z|^N - NA|z|^{N-1}.$$

Therefore

$$\left|\frac{NP - zP'}{zP}\right| \leqq \frac{N^2A}{|z|(|z| - NA)}$$

if $|z| > \max(1, NA)$. Hence

$$|S_n((NP - zP')/zP; 0, a)| \leqq \frac{8aN^2A}{a(a - NA)} = \frac{8N^2A}{a - NA}.$$

Therefore

$$4N < |2\pi_n Ni| < \frac{K_1}{m^3 n} + \frac{8N^2A}{a - NA},$$

and

$$m^3 < \frac{K_1(a - NA)}{4Nn(a - 3NA)} \leqq \frac{K_1}{2Nn} = \frac{K}{n}.$$

Theorem 2 is already sufficient for our purpose. We mention, however, that a much better estimate for m immediately follows from Theorem 2.

COROLLARY 1. *If $a \geqq 5NA$, and $\epsilon > 0$, then there is a point z_ϵ inside $C_a(0)$ at which $|P(z_\epsilon)| < \epsilon$.*

We need only take $n > K/\epsilon^3$ in Theorem 2.

The numbers $P((k+il)a/n)$ can be calculated in a finite number of steps by rational operations only, and therefore the number z_ϵ in the above corollary can be found constructively. Since the points $(k+il)a/n$ for a given n can be ordered, say by letting $(k_1+il_1)a/n$ precede $(k_2+il_2)a/n$ if $k_1 < k_2$ or $k_1 = k_2$ and $l_1 < l_2$, then if for a given n we always choose the first such point which satisfies the inequality $|P(z)| < \epsilon$, no element of arbitrary choice is involved.

5. The fundamental theorem of algebra. Theorem 2 is the crux of this paper; from here on we can proceed in many ways. The following argument was suggested by the proof of Brouwer and deLoor but is perhaps more direct.

LEMMA 3. *Let $a_N = 1$, and $0 < \epsilon < 1$. Then we can find points $z_1, \cdots, z_N$ such that*

$$|P(z_i)| < \epsilon, \qquad i = 1, \cdots, N,$$

and such that if $|P(z)| < \delta$, where $\epsilon \leqq \delta < 1$, then

$$\min_{1 \leqq i \leqq N} |z_i - z| < 2\delta^{1/2^N}.$$

Proof. We shall actually prove the lemma with the factor $2^{1-1/2^N}$ instead of 2. First we note that if $|z| \geqq 1 + NA$, then

$$|P(z)| \geqq |z|^N - NA\,|z|^{N-1} = |z|^{N-1}(|z| - NA) > 1. \tag{3}$$

The lemma is obvious if $N = 1$. Suppose that it is true for $N-1$. Let $0 < \epsilon_1 < \epsilon$, and by Theorem 2 we find a point z_1 such that $|P(z_1)| < \epsilon_1$. By (3), $|z_1| < 1 + NA$. Now

$$P(z) = P(z_1) + (z - z_1)P_1(z),$$

where

$$P_1(z) = \sum_{m=0}^{N-1} b_m z^m,$$

and

$$|b_m| = \left| \sum_{k=m+1}^{N} a_k z_1^{k-1-m} \right| < NA(1 + NA)^{N-1} = A_1.$$

By assumption we can find points $z_2, \cdots, z_N$ such that $|P_1(z_i)| < \epsilon_1$, $i = 2, \cdots, N$, and such that

$$|P_1(z)| < \delta, \quad \text{where} \quad \epsilon_1 \leqq \delta < 1,$$

implies

$$\min_{2 \leq i \leq N} |z_i - z| < 2^{1-1/2^{N-1}} \delta^{1/2^{N-1}}.$$

By (3) applied to $|P_1(z)|$, we have $|z_i| < 1 + (N-1)A_1$, $i = 2, \cdots, N$. Hence

$$|P(z_i)| < \epsilon_1 + |z_i - z_1| \epsilon_1 < C\epsilon_1,$$

where

$$C = 1 + 1 + (N - 1)A_1 + 1 + NA.$$

Now if $|P(z)| < \delta$, then

$$|(z - z_1)P_1(z)| < \delta + \epsilon_1 \leqq 2\delta.$$

Therefore either

$$|z - z_1| < (2\delta)^{1/2} < 2^{1-1/2^N} \delta^{1/2^N}, \tag{4}$$

or

$$|P_1(z)| < (2\delta)^{1/2}.$$

In the latter case,

$$\min_{2 \leq i \leq N} |z - z_i| < 2^{1-1/2^{N-1}} (2\delta)^{1/2^N}. \tag{5}$$

The required inequality follows from (4) and (5). If we choose $\epsilon_1 = \epsilon/C$, all the assertions in the lemma follow.

THEOREM 3. *We can find a convergent sequence* $\{z_n\}$ *such that* $P(z_n) \to 0$.

Proof. Let $\epsilon_n = 2^{-n2^N}$, $n = 1, 2, \cdots$. By Lemma 3 we can find points $z_{1n}, \cdots, z_{Nn}$, such that $|P(z_{\nu n})| < \epsilon_n$, $\nu = 1, \cdots, N$, and such that if $|P(z)| < \delta$, where $\epsilon_n \leqq \delta < 1$, then

$$\min_{1\leq\nu\leq N} |z - z_{\nu n}| < 2\delta^{1/2^N}.$$

Let $z_1 = z_{11}$. If z_n has already been chosen as one of the points $z_{\nu n}$, $\nu = 1, \cdots, N$, then

$$|P(z_n)| < \epsilon_n, \tag{6}$$

and $\epsilon_{n+1} \leqq \epsilon_n < 1$. Hence there is a point $z_{\mu,n+1}$ such that

$$|z_n - z_{\mu,n+1}| < 2\epsilon_n^{1/2^N} = 2^{1-n}.$$

We choose z_{n+1} to be that one for which μ is smallest. Hence we have constructed the sequence $\{z_n\}$ in such a way that

$$|z_n - z_{n+1}| < 2^{1-n},$$

and, by (6), $P(z_n) \to 0$. It follows immediately that $\{z_n\}$ is convergent, for

$$\begin{aligned} |z_n - z_{n+p}| &\leqq |z_n - z_{n+1}| + \cdots + |z_{n+p-1} - z_{n+p}| \\ &< 2^{1-n} + \cdots + 2^{1-(n+p-1)} < 2^{2-n}, \end{aligned}$$

which can be made arbitrarily small.

6. Another approach. Theorem 3 is, of course, the fundamental theorem of algebra. We believe that it is worth while to give another approach which may be more desirable for, say, pedagogical purposes. If $P(z)$ is relatively prime to $P'(z)$, that is, if the discriminant of $P(z)$ is different from zero, we can construct a root by Newton's method of approximation. Since Newton's method is not usually discussed in the complex domain, we shall, for the sake of completeness, give a proof of its validity. This proof is essentially the same as that of Brouwer and deLoor, except that we carry the computations out in more detail in order to get explicit (but crude) estimates for the constants involved.

Let $P(z)$ be relatively prime to $P'(z)$. Then we can find by a finite number of rational operations (see [4], p. 91) polynomials $C(z)$ and $D(z)$ of degree at most N satisfying

$$C(z)P(z) + D(z)P'(z) \equiv 1.$$

Let $C(z) = \sum_{k=0}^{N} c_k z^k$, $D(z) = \sum_{k=0}^{N} d_k z^k$, $C = \max|c_k|$, and $D = \max|d_k|$; let $E = NA + 1$, and let $a_N = 1$. We can assume, of course, that $N \geqq 2$.

LEMMA 4. *If*

$$|P(z)| < \alpha = \min\left(\frac{1}{2NCE^N}, \frac{1}{4NDE^N}, 1, \frac{\lambda^2}{g}\right), \text{ where } \lambda = \frac{1}{2NDE^N} \text{ and}$$

$$g = 2\gamma AE^N,$$

then

$$|P'(z)| > \lambda,$$

and

$$\left|P\left(z - \frac{P(z)}{P'(z)}\right)\right| < \frac{g}{\lambda^2}|P(z)|^2 < |P(z)|.$$

Proof. By (3), we have $|z| < E$. Then $|C(z)| < NCE^N$, $|D(z)| < NDE^N$, and

$$1 < NCE^N\alpha + NDE^N|P'(z)|,$$

so that

$$|P'(z)| > \lambda.$$

Let $z_1 = z - P(z)/P'(z) = z + h$; then

$$|h| = \left|-\frac{P(z)}{P'(z)}\right| < \frac{|P(z)|}{\lambda} < \frac{1}{2}.$$

Now

$$P(z_1) = P(z) + hP'(z) + \sum_{k=2}^{N} h^k \frac{P^{(k)}(z)}{k!}$$

$$= h^2 \sum_{k=2}^{N} h^{k-2} \frac{P^{(k)}(z)}{k!}.$$

By (2),

$$\left|\frac{P^{(k)}(z)}{k!}\right| < \gamma A E^N,$$

and therefore

$$|P(z_1)| < |h|^2 \sum_{k=0}^{N-2} |h|^k \gamma A E^N < 2\gamma A E^N |h|^2 < \frac{g}{\lambda^2}|P(z)|^2 < |P(z)|.$$

THEOREM 4. *If* $|P(z_1)| < \alpha$, *then the sequence* $\{z_n\}$ *defined by*

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}, \qquad n = 1, 2, \cdots,$$

is convergent and $P(z_n) \to 0$.

Proof. Let

$$w_n = P(z_n), \qquad h_n = -\frac{P(z_n)}{P'(z_n)} = z_{n+1} - z_n, \qquad n = 1, 2, \cdots.$$

Then we can easily show by induction from Lemma 4 that

$$|w_n| < \frac{\lambda^2}{g}\theta^{2^n-1}, \tag{7}$$

and

$$|h_n| < \frac{\lambda}{g}\theta^{2^n-1}$$

where

$$\theta = \frac{g|w_1|}{\lambda^2} < 1 .$$

Then

$$\begin{aligned} |z_{n+p} - z_n| &= |h_n + \cdots + h_{n+p-1}| \\ &< \frac{\lambda}{g}(\theta^{2^n-1} + \cdots + \theta^{2^{n+p-2}}) \\ &< \frac{\lambda\theta^{2^n-1}}{g(1-\theta)}, \end{aligned}$$

which shows that $\{z_n\}$ satisfies the Cauchy convergence criterion. The rest of the theorem follows from (7).

If the Euclidean algorithm can be performed to obtain the greatest common divisor of $P(z)$ and $P'(z)$, then even if the discriminant of $P(z)$ vanishes, we can construct a root of $P(z)$ by Newton's method. For if $P(z)$ is divided by this greatest common divisor, we obtain a polynomial which is relatively prime to its derivative, to which Theorem 4 applies. In general, however, the Euclidean algorithm cannot be performed in a finite number of steps since to carry out the second division we must know the degree of the remainder in the first division; this requires that we know the highest power of z whose coefficient is different from zero. But in general it is impossible to determine constructively in a finite number of steps whether a given real number is zero or not. If the coefficients are in a field like that of the rational numbers or of the algebraic numbers where such a constructive process always exists, Newton's method can always be used. We make a final remark that the application of Theorem 4 depends on Theorem 2 which shows how a point z_1 satisfying $|P(z_1)| < \alpha$ can be found.

Bibliography

1. L. E. J. Brouwer and B. deLoor, Intuitionistischer Beweis des Fundamentalsatzes der Algebra, Amsterdam Kon. Akad. van Wetenschappen, Proc., vol. 27, 1924, pp. 186–188.

2. L. E. J. Brouwer, Intuitionistische Ergänzung des Fundamentalsatzes der Algebra, *ibid.*, pp. 631–634.

3. K. Knopp, Funktionentheorie, Erster Teil, Leipzig, 1937. English translation by F. Bagemihl. New York, Dover, 1945.

4. B. L. Van der Waerden, Moderne Algebra, Berlin, 1937.

5. H. Weyl, Randbemerkungen zu Hauptproblemen der Mathematik, Math. Zeit., vol. 20, 1924, pp. 131–150.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE PROOF OF FEUERBACH'S THEOREM

H. F. SANDHAM, Trinity College, Dublin, Ireland

Let A, B, C denote the vertices of a triangle, a, b, c the lengths of the opposite sides whose mid points are L, M, N, and X, Y, Z the points of contact of these sides with the inscribed circle.

It is readily proved

$$MN = a/2, \qquad NL = b/2, \qquad LM = c/2,$$
$$LX = \pm (b - c)/2, \qquad MY = \pm (c - a)/2, \qquad NZ = \pm (a - b)/2,$$

and therefore

$$\pm LX \cdot MN \pm MY \cdot NL \pm NZ \cdot LM = 0. \tag{1}$$

Hence, by the converse of Ptolemy's theorem, there is a point P concyclic with L, M, N whose distances from these points are proportional to LX, MY, NZ—the tangents from L, M, N to the inscribed circle. Since the locus of points from which the tangents to two circles are in a constant ratio is a coaxal circle, therefore the circumcircle of L, M, N is coaxal with the inscribed circle and the point circle P. That is P is a limiting point, and since it lies on one of them, the circumcircle of L, M, N and the inscribed circle therefore touch at P.

The relation (1) holds if X, Y, Z are the points of contact of an escribed circle and Feuerbach's theorem is thus proved for this case also.

HYPERSPACIAL FIGURATE PROGRESSIONS

WILLIAM FUNKENBUSCH, Michigan College of Mining and Technology

We shall say that a set of numbers forms a figurate progression if they are the h to $(h+k)$th figurate numbers inclusive for a constant dimension n and a constant linear difference d. An arithmetic progression is then a linear figurate progression. Having recently developed a method for finding products of numbers in arithmetical progression by logarithms, I looked at the figurate progression for a generalization of the procedure.

The mth figurate number of dimension n and difference d is given by

$$M_{n,d} = \frac{1}{n!} \frac{(m + n - 2)!}{(m - 1)!} [(m - 1)d + n].$$

Of course for linear figurate numbers (numbers in arithmetical progression) we obtain

$$M_{1,d} = (m-1)d + 1.$$

We also notice that the sum of the first m terms of the progression gives the mth figurate number of dimension $(n+1)$ and difference d:

$$M_{n+1,d} = \sum_{d=1}^{m} M_{n,d}.$$

The product of the h to the $(h+k)$th terms inclusive of an n-space figurate progression of linear difference d is given by

$$(1) \qquad {}_{h,k}F_{n,d} = \prod_{m=h}^{h+k} \frac{1}{n!} \frac{(m+n-2)!}{(m-1)!} [(m-1)d + n].$$

Let us now introduce bifactorial a,

$$a!! = 1!2!3! \cdots a!.$$

By use of bifactorial notation, (1) may be written

$$(2) \qquad {}_{h,k}F_{n,d} = \left(\frac{1}{n!}\right)^{k+1} \frac{(h+k+n-2)!!}{(h+n-3)!!} \cdot \frac{(h-2)!!}{(h+k-1)!!} [{}_{h,k}A_{n,d}].$$

where ${}_{h,k}A_{n,d}$ is the arithmetical progression product

$${}_{h,k}A_{n,d} = [(h-1)d + n][hd + n][(h+1)d + n] \cdots [(h+k-1)d + n].$$

We may write

$${}_{h,k}A_{n,d} = d^{k+1} \frac{(h-1)d + n}{d} \cdot \frac{hd + n}{d} \cdot \frac{(h+1)d + n}{d} \cdots \frac{(h+k-1)d + n}{d}$$

and by use of fractional factorials as defined by the author*

$$\left(\frac{b}{c}\right)! = \frac{b}{c}\left(\frac{b}{c} - 1\right)!, \quad b \text{ and } c \text{ positive and rational, } b > c,$$

$$\left(\frac{b}{c}\right)! = \frac{b}{c}, \qquad b \text{ and } c \text{ positive and rational, } b < c,$$

$$(3) \qquad {}_{h,k}A_{n,d} = d^{k+1} \frac{\left[\frac{(h+k-1)d + n}{d}\right]!}{\left[\frac{(h-2)d + n}{d}\right]!}.$$

It is of course apparent that

* The λ Function, copyright 1944.

$$ {}_{h,k}F_{1,d} = {}_{h,k}A_{1,d}. $$

By using (3) in (2) we get the desired formula

$$ (4) \qquad {}_{h,k}F_{n,d} = \left[\frac{d}{n!}\right]^{k+1} \frac{(h+k+n-2)!!(h-2)!!\left[\dfrac{(h+k-1)d+n}{d}\right]!}{(h+n-3)!!(h+k-1)!!\left[\dfrac{(h-2)d+n}{d}\right]!}. $$

The F's obviously lend themselves to logarithmic calculation. The author possesses the only tables of logs of bifactorial and fractional factorials of which he knows.

Example. Obtain the product of the 2nd to the 5th four dimensional figurate numbers inclusive, of linear difference 5. These numbers are of course 9, 35, 95 and 210 and their product is 6,284,250. The logarithmic solution here of course is just to illustrate method.

Solution:

$$ {}_{2,5}F_{4,5} = \left[\frac{5}{4!}\right]^4 \frac{7!!0!!\left(\dfrac{24}{5}\right)!}{3!!4!!\left(\dfrac{4}{5}\right)!} $$

$$ \begin{aligned} \log 5 &= 10.69897 - 10 \\ -\log 4! &= \underline{1.38021} \\ & \quad 9.31876 - 10 \\ & \quad \underline{\qquad\qquad 4} \\ \log\left[\frac{5}{4!}\right]^4 &= 7.27504 - 10 \end{aligned} $$

$$ \begin{aligned} \log 7!! &= 11.09833 \\ \log 0!! &= 0.00000 \\ \log\left(\frac{24}{5}\right)! &= \underline{1.86655} \\ \log \text{Num.} &= 10.23992 \\ -\log \text{Den.} &= \underline{3.44166} \\ \log {}_{2,5}F_{4,5} &= 6.79826 \\ {}_{2,5}F_{4,5} &= 6,284,300 \end{aligned} $$

$$ \begin{aligned} \log 3!! &= 1.07918 \\ \log 4!! &= 2.45939 \\ \log\left(\frac{4}{5}\right)! &= \underline{9.90309 - 10} \\ \log \text{Den.} &= 3.44166 \end{aligned} $$

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Engineering Preview. By L. E. Grinter, H. N. Holmes, H. C. Spencer, Rufus Oldenburger, Charles Harris, R. G. Kloeffler, and V. M. Faires. New York, Macmillan Co., 1945. 10+581 pages. $4.50.

Engineering Preview is designed to orient the reader in fields of science and engineering. Each of seven authors has contributed a section of the book in which each writer discusses the tools of the engineer, mathematics, physics, chemistry and engineering drawing.

The headings and authors of the chapters are: (1) Outlines of Science and Engineering—Orientation, by L. E. Grinter; (2) Chemistry—The Science of Matter and Molecules, by H. N. Holmes; (3) Technical Drawing—The Language of the Engineer, by H. C. Spencer; (4) Mathematics—A Universal Tool of Engineering, by R. Oldenburger; (5) The Slide Rule—The Engineer's Coat of Arms, by C. O. Harris; (6) Light and Electricity—Hallmarks of Civilization, by R. G. Kloeffler; and (7) Mechanics and Thermodynamics, by V. M. Faires. Dr. Holmes is from Oberlin College, Ohio; Professors Kloeffler and Faires from Kansas State College and Texas A. and M. College respectively and the remaining writers are from the Illinois Institute of Technology. The book appends fifty pages of tables adapted from the Macmillan Tables.

The first section is devoted entirely to orientation and includes a simple self scoring "parlor test" giving the reader a rough measure of his technological traits. Roughly one hundred pages are devoted to rapid surveys of subject matter in each of the sections on chemistry, drawing, and mathematics. It is not intended that this survey is to serve as an initial textbook even though a large number of problems is included. If the reader is partially familiar with some of the material, *Engineering Preview* provides him with a stimulating review of engineering technology. A diligent student who gives deliberate attention to details and solves all the problems will find himself admirably equipped to enter a technological curriculum.

In the opinion of this reviewer, the first chapter on orientation should be read by all freshmen engineers to bring into focus the fields of engineering and their relationships to the basic sciences. The chapter on drawing is a gem as are all the figures in the whole volume. The reviewer disagrees with the exposition and figures relating to a unit solid angle on page 400.

D. L. HOLL

How To Solve It. A New Aspect Of Mathematical Method. By G. Polya. Princeton University Press, 1945. 15+204 pages. $2.50.

The interest of this book is pedagogical. First it should be noticed that Professor Polya explicity disavows the word "new" in the subtitle. As he observes in his preface, "heuristic . . . has a long past." Probably 2300 years would not be an excessive estimate. Then there is the statement on the jacket, "*A system of thinking which can help you solve any problem.*" Anyone who has been so rash as to write a book for what commercial publishers call "the trade," will know that an author is not responsible for the enthusiasms of his publishers, who must "make friends and influence people," or go out of business. Only those mathematicians who refrain from introspection and from observing their colleagues, might believe that a facility in solving problems in elementary mathematics will help them to "solve any problem." It may be suspected that Professor Polya did not write the copy for the jacket of his book.

What he did write, is an instructive exposition of the heuristic method applied to the solution of problems in elementary mathematics. Not to delude the reader into expecting more than can be offered, Professor Polya states (p. 158) that "Infallible rules of discovery leading to the solution of all possible mathematical problems would be more desirable than the philosopher's stone, vainly sought by all alchemists." And, quite bluntly: "The first rule of discovery is to have brains and good luck. The second rule of discovery is to sit tight and wait till you get a bright idea." Those of us who have little luck and less brains sometimes sit for decades. The fact seems to be, as Poincaré observed, it is the man, not the method, that solves a problem.

So far as instruction is concerned, the tactics of problem solving as expounded here are probably better known to teachers of secondary-school mathematics than they are to a majority of university professors. A generation ago, courses in the pedagogy of mathematics for prospective teachers in American secondary schools included substantially the subject-matter of this book. Such may still be the case; only those having direct contact with the teaching of teachers will know. If heuristic is no longer taught, *How To Solve It* may supply the deficiency. Every prospective teacher should read it. In particular, graduate students who are required to do some teaching, will find it invaluable if they have not already profited by observing one of their own teachers—if they were lucky enough to have one—who knew how to teach. "The traditional mathematics professor" (p. 181) who reads a paper before one of the Mathematical Societies, might also learn something from the book: "He writes a, he says b, he means c; but it should be d."

E. T. Bell

Sampling Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., 1945. 12+498 pages. $4.00.

This is the second of two volumes written under the joint title, *Fundamentals of the Theory of Statistics*. The previously published first volume, *Elementary Statistics and Applications*, has been reviewed in a recent issue of the MONTHLY. A 22-page introduction sets forth briefly those definitions and basic concepts of the first volume which are essential to the present one. Thus it can be read independently of the first, particularly by advanced students or research workers.

The body of the book has three main divisions: general theory of frequency curves, elementary theory of random sampling, and advanced sampling problems. The first of these begins with a discussion of probability and the symmetric binomial distribution along the same lines as in the first volume, proceeds to the asymmetric binomial and hypergeometric distribution and finally makes the transition to the various Pearson curves. The principal line of development is descriptive in nature with generous use of numerical examples and diagrams while the logical derivations are collected in a mathematical appendix immediately following. Then comes an excellent discussion of the Gram-Charlier type A series, cumulants and all, presented according to the same pedagogical pattern. Noteworthy is a short lucid comparison of the assumptions underlying these two approaches. There is a summary of conditions leading to normality and to non-normality and three outstanding non-normal distributions, t, χ^2, and F are defined and illustrated. A full chapter is devoted to numerical calculations for frequency curves, involving explanation of tables, plotting, computation of areas, curve fitting, and χ^2 test.

Part II presents first a comprehensive preview of sampling theory. This includes general remarks on the technique of random sampling and an unusually precise discussion of statistical inference, the testing of hypotheses, determination of confidence intervals, and maximum likelihood estimation of population parameters. Stratified and purposive sampling are mentioned briefly. Then these general aspects of sampling procedure are illustrated by a detailed study of the simplest case, samples from a discrete two-fold population. The Poisson distribution also gets treatment here. Next in order come samples from a normal population and these receive the most extensive discussion of the book. Samples of size two are used for the most part. This puts the usual multidimensional reasoning into the plane where a remarkably clear presentation is effected by means of elaborate tables and diagrams and a minimum of calculus. The distribution of the sample mean, variance, and t statistic are the end results. The N-dimensional case is also treated. In the testing of hypotheses regarding sample mean and variance a point is strongly made for the occasional use of unsymmetric regions of rejection. The distinction between the cases of known and estimated population variance is brought out with some unusual diagrams. The distribution of correlation coefficients, ordinary, partial, and multiple, are related to the normal and F distributions without proofs but with many illustrative problems.

The final division of the book opens with the multinomial distribution leading up to a clever and admittedly non-rigorous derivation of χ^2. Examples illustrate several different applications of the latter, particularly the testing of independence of classification. Quite novel is the joint study of mean and standard deviation; large tables illustrate the many possible areas of acceptance in joint significance tests. Also included is a treatment of sampling fluctuations in regression coefficients and higher order variances in any number of dimensions. Just about every question that can be raised in connection with pairs of samples is discussed: difference between percentages, means, variances, correlation coefficients, and regression statistics. Through these latter portions of the book proofs are generally omitted as being beyond its mathematical scope. However plenty of real-life examples are worked through to illustrate procedures. The chapter on analysis of variance is particularly instructive. Without developing highly specialized techniques the various fundamental procedures are presented through successive analyses of a set of student grades into more and more component variances. A short meaty concluding chapter, rich in references, introduces the reader to the difficulties of the problem of non-normality.

There is a subject and author index and an appendix with 12 mathematical tables including t, χ^2, F, and ratio of sample mean deviation to standard deviation. Scattered through the book are 113 figures, 56 tables, and footnote references to over 70 books and papers. The lack of exercises would seem to limit its use in the classroom. A number of irritating misprints and misstatements indicate some careless proofreading. However none of these, except possibly footnote 2 on page 117, should be permanently misleading.

The authors are economists, not professional mathematicians, and have bravely set about writing a book which was to be rational, comprehensive, even scholarly, yet be completely understandable to a reader having meagre acquaintance with the calculus. In pursuit of this last aim they sometimes become repetitious, sometimes confuse with excessive particulars, but more often they succeed in giving the reader a better working knowledge of the material than would a more concise and sophisticated presentation. Many of the artifices used to circumvent relatively difficult mathematics cannot help but clarify the subject, even to the initiated, by opening up new viewpoints. Some of these seem of general pedagogical importance. The book should help destroy the vacuum between purely descriptive texts on advanced statistical methods and formal treatises on the advanced mathematical theory.

J. L. VANDERSLICE

NEW BOOKS RECEIVED

Plane and Spherical Trigonometry. Second Edition. By H. A. Simmons. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1945. 11+387 pages. $2.25.

The River Mathematics. By A. Hooper. New York, Henry Holt and Co., 1945. 8+401 pages. $3.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 687. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

E 696. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Find four positive integers a, b, c, d such that a, b, c are in geometric progression, b, c, d in arithmetic progression, and $c+d=44$.

E 697. *Proposed by C. A. Murray, West Texas State College*

A certain geometry text raises the question whether the following procedure will inscribe a regular n-gon in a circle: AB being a diameter of the circle, construct an equilateral triangle ABC. Divide AB into n equal parts and let D be the second point of division from A. Draw CD, producing it to cut the circle at E. Is AE the side of a regular n-gon inscribed in the circle? For n equal to 3 or 4 the answer is readily affirmative. Does the procedure yield a regular pentagon for n equal to 5? If not, give a measure of the error.

E 698. *Proposed by J. M. Feld, Brooklyn College*

Let the complex numbers a and b represent two points on the Gauss plane. If $\rho=re^{i\alpha}$, (r and α real), show how one can construct the point corresponding to $z=(a+\rho b)/(1+\rho)$.

E 699. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 be the feet of the altitudes, the symmedians, and the cevians through the circumcenter, on the sides BC, CA, AB of a triangle ABC. (a) The lines B_1C_1, B_2C_2, B_3C_3 are concurrent in a point M_1; C_1A_1, C_2A_2, C_3A_3 in a point M_2; A_1B_1, A_2B_2, A_3B_3 in a point M_3. (b) Triangle $M_1M_2M_3$ is homological to, and circumscribes, triangle ABC.

E 700. *Proposed by Arnold Dresden, Swarthmore College*

If h and i are respectively positive and non-negative integers, then

(a) $$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_0, p_1, \cdots, p_i$ which satisfy the conditions $p_0+p_1+\cdots+p_i=h$, $p_1+2p_2+\cdots+ip_i=i$;

(b) under the further restriction $i \geqq h-1$,

$$\sum \frac{1}{p_1!\cdots p_{i+1}!} = \frac{i!}{(i+1-h)!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_1, \cdots, p_{i+1}$ for which $p_1+\cdots+p_{i+1}=h$, $p_1+2p_2+\cdots+(i+1)p_{i+1}=i+1$.

SOLUTIONS

Self-reciprocal conics

E 659 [1945, 95]. *Proposed by R. A. Staal, University of Toronto*

Show that, if one conic is self-reciprocal with respect to another, then the two conics belong to a symmetrical set of four, each of which is self-reciprocal with respect to any of the other three. (However, not more than three of the four conics can be real.)

Solution by J. A. Jenkins, Loganton, Pa. The self-reciprocal property implies that the two polarities (say P and Q) associated with the conics are permutable. Thus the collineation obtained by performing them in turn is of period two, and so a harmonic homology (say H). Moreover, it is clear that the center and axis of this homology are pole and polar with respect to both P and Q. Each of these latter is seen to induce the same involution in the axis of the homology, since its points are invariant under their product. Choosing any pair of points in this involution we see that they form, with the center of the homology, the vertices of a triangle self-polar with respect to each of the given conics. Let us call the harmonic homologies, having as centers and axes the vertices and opposite sides of this triangle, H (as before) and K and L. The product correlations of these with P will be the polarity Q and two new polarities R and S. H, K, L, P, Q, R, S and the identical collineation form an abelian group of order eight. In particular, the conics associated with P, Q, R, and S are the required four conics. (Indeed we see that there is an infinity of such sets.) If the first two are real it is necessary that the pair of points chosen above be real. In this case, by examination of the corresponding involutions induced in the sides of the common self-polar triangle, we see that one of the associated conics will be real and the other not. This completes the proof.

A Tetrahedron of Constant Volume

E 667 [1945, 218]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let P, Q, R, P', Q', R' be arbitrary points on the respective edges BC, CA, AB, DA, DB, DC of a given tetrahedron $ABCD$. Prove that four planes, parallel to the faces BCD, CDA, DAB, ABC, drawn through the centroids of the respective tetrahedra $AQRP'$, $BRPQ'$, $CPQR'$, $DP'Q'R'$, form a tetrahedron of constant volume.

Solution by the Proposer. Let G, G', G'' be the mean centers (for equal multiples) of the sets of points $(A, B, C, D, P, Q, R, P', Q', R')$, (A, B, C, P, Q, R), (D, P', Q', R'). Now G'' is the centroid of tetrahedron $DP'Q'R'$, G' lies in the plane ABC, and G, G', G'' are collinear with $GG'/GG'' = -2/3$. Similar statements may be made when any other vertex of the given tetrahedron is favored. We thus see that the final tetrahedron is inversely homothetic to the given tetrahedron, having G for homothetic center, and $-2/3$ for homothetic ratio. The volume of the final tetrahedron is then $8/27$ that of the given tetrahedron.

The analogous theorem and proof hold for the plane.

A Diophantine Equation

E 668 [1945, 218]. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Prove that the equation $x^2 - 3y^2 = 17$ has no solution in integers.

I. *Solution by E. P. Starke, Rutgers University.* Any integer x takes one of the forms $3n$, $3n \pm 1$. If these are substituted into the proposed equation, the results may be written as

$$3(3n^2 - y^2) = 17, \qquad 3(3n^2 \pm 2n - y^2) = 16,$$

respectively. These are evidently impossible in integers.

II. *Solution by W. F. Cheney, Jr., University of Connecticut.* The given equation demands that $x^2 \equiv 17 \equiv 2$, mod 3. But 2 is a quadratic non-residue, mod 3.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, H. N. Carleton, J. M. Danskin, Roy Dubisch, R. L. Duncan, Daniel Finkel, Irving Kaplansky, H. L. Lee, Benjamin Liebowitz, Irma Moses, W. J. Robinson, E. D. Schell, W. H. Thompson, and the proposer. Several solvers noted that 17 could be replaced by any integer of the form $2+3k$.

Editorial Note. The problem is also an immediate consequence of the theorem: *The diophantine equation $x^2 - Cy^2 = \pm H$, where C is positive and not a perfect square and H is positive and $> \sqrt{C}$, has no solution if no positive integer $K \geqslant \frac{1}{2}H$ can be found which makes $(K^2 - C)/H$ integral.* By setting $K = 1, 2, \cdots, 8$ in turn we find that $(K^2 - 3)/17$ is in no case an integer.

The above theorem is established on pages 482 and 483 of the second volume of Chrystal's *Algebra*. In arts. 15 through 19 of chapter XXXIII of this book Chrystal, using the theory of continued fractions, gives a complete discussion of integral solutions of quadratic forms of the type involved in this problem. For treatments not employing continued fractions see chapter XI (especially art. 6) of *Elementary Number Theory* by Uspensky and Heaslet, or chapter IV of *Modern Elementary Theory of Numbers* by Dickson.

The Centroid of n Points

E 669 [1945, 218]. *Proposed by J. H. Butchart, Grinnell College*

Let G be the centroid of n coplanar points P_i, Q any point in the same plane, and Δ_i the signed area of the triangle QGP_i. Show that

$$\Sigma\Delta_i = 0.$$

I. *Solution by Janet Ryder, Grinnell College.* Let rectangular coordinates of G, P_i, and Q be respectively $(\bar{x}, \bar{y})$, (x_i, y_i), and (x, y). Then $n\bar{x} = \Sigma x_i$, $n\bar{y} = \Sigma y_i$, and

$$\Sigma\Delta_i = \tfrac{1}{2}\Sigma \begin{vmatrix} x & y & 1 \\ \bar{x} & \bar{y} & 1 \\ x_i & y_i & 1 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} x & y & 1 \\ \bar{x} & \bar{y} & 1 \\ \Sigma x_i & \Sigma y_i & n \end{vmatrix} = 0.$$

II. *Solution by C. E. Springer, University of Oklahoma.* If the centroid G is taken as origin of a set of n vectors V_i to the points P_i, then $\Sigma V_i = 0$. Denoting the vector GQ by V, the area Δ_i is one-half the scalar magnitude of the cross product of V and V_i. Now $\Sigma(V \times V_i) = V \times (\Sigma V_i) = 0$. Hence $\Sigma\Delta_i = 0$.

III. *Solution by L. M. Kelly, U. S. Coast Guard Academy.* It is well known that the algebraic sum of the perpendiculars dropped from any number of points P_i onto a line passing through the centroid G of these points is equal to zero. Taking the line as GQ, this algebraic sum times the common base GQ gives twice the algebraic sum of the areas Δ_i, and is zero.

That the algebraic sum of the perpendiculars from P_i onto GQ is zero may be seen at once by imagining unit masses placed at the points and then taking moments about GQ.

Also solved by J. M. Danskin, Sydney Glusman, Francis Hall, Irving Kaplansky, E. D. Shell (all like solution I), the proposer (like solution II), and S. T. Parker. The proposer noted that if the signed areas are represented by axial vectors, then the points P_i need not be coplanar. S. T. Parker solved, in a manner analogous to solution I, the more general problem: *Let G be the centroid of n coplanar points P_i, Q and R any two points in the same plane, Δ_i the signed area of the triangle QRP_i, and Δ the signed area of the triangle QRG. Then* $\Sigma\Delta_i = n\Delta$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4181. *Proposed by P. D. Thomas, Lumberton, Miss.*

Lines are drawn from a point P on the circumcircle of an equilateral triangle parallel to the three sides thus determining six points, two on each side respectively. (1) Prove that the six points thus determined lie by threes on two straight lines. (2) If Q is the point of intersection of these two lines, find the locus of Q as P moves on the circumcircle.

4182. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Show that the envelope of the conics circumscribing a given triangle and such that the angle between the asymptotes is constant is a curve of the fourth degree bitangent to the line at infinity at the circular points and having the vertices of the triangle for double points.

4183. *Proposed by P. M. Hummel, University of Alabama*

Let p and q be non-negative integers and x a variable. Define

$$f(x, p, q) = \sum_{i=0}^{p} (-1)^i {}_pC_i (x - i)^q,$$

where ${}_pC_i$ are binomial coefficients. Prove that $f(x, p, q)$ equals zero if $p > q$; equals $p!$ if $p = q$; and is a polynomial in x of degree $q - p$ if $p < q$.

4184. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what systems of numbers with the base less than 10000 are there the greatest number of squares of four digits of the form $aabb = (cc)^2$? Dedicated to E. P. Starke.

SOLUTIONS

Matrix Equation

4126 [1944, 352]. *Proposed by A. D. Wallace, University of Pennsylvania*

Let x, A, b denote respectively $(1, m)$, (m, n), $(1, n)$ matrices, an (i, j) matrix being one with i rows and j columns. If the matrix AA' is non-singular, show that the least square solution of $xA = b$ is the solution of an equation of the form $xA = b_0$. Determine b_0 and its geometric meaning.

Solution by the Proposer. Let $T(x)=xA$ so that T is a linear transformation taking E_m into E_n, euclidean spaces of dimensions m and n. Here we interpret a $(1, p)$ matrix as a point in E_p. Since AA' is non-singular it is readily seen that M, the transform of E_m under T, is of dimension m. While b is in E_n it will not in general lie in M. Let $\delta(x)=(xA-b)(xA-b)'$. Then $\delta(x)$ is the square of the distance from the point xA of M to the point b. It is also the sum of the squares of the deviations of $xA=b$. We assert that if $x_0AA'-bA'=0$ then $\delta(x_0)$ is an absolute minimum. For

$$\delta(x_0+y)=\delta(x_0)+2(x_0AA'-bA')y'+yAA'y',\ y \text{ any } (1, n) \text{ matrix.}$$

The second term vanishes and the third is $(yA)(yA)'$. This latter is a sum of squares and so non-negative. Further if $yA=0$ then $yAA'=0$ and thus $y=0$ since AA' has an inverse. We conclude that $\delta(x)$ is greater than $\delta(x_0)$ for any x different from x_0. If we multiply $x_0AA'-bA'=0$ on the right by $(AA')^{-1}$ we see that $x_0=bA'(AA')^{-1}$ and this latter is the least square solution of $xA=b$. The transform of x_0, $b_0=x_0A$, lies in M and is the nearest point of M to b. Now $x_0A=bA'(AA')^{-1}A=b_0$. Hence x_0 is a solution of $xA=bA'(AA')^{-1}A$. It is readily seen that the equation $xA=b_0$ has a unique solution.

Tetrahedron, Equal Powers

4131 [1944, 409]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $ABCD$ be a skew quadrangle; planes perpendicular at A to AC, at C to CB, at B to BD, at D to DA form a tetrahedron $A_1C_1B_1D_1$ with the centroid G_1. Show that the powers of A with respect to the sphere (G_1C) on G_1C as diameter, of C with respect to (G_1B), of B with respect to (G_1D), of D with respect to (G_1A) are equal.

Solution by Howard Eves, College of Puget Sound. We shall employ the following lemma—a theorem which is interesting in itself.

LEMMA. *Consider a tetrahedron T and a directed polygonal line p of four segments in space. If each segment of p is perpendicular to a face of T, is directed from the outside to the inside of this face, and is proportional to the area of this face, then p is a closed space polygon (skew-quadrangle). Conversely, if p is a skew-quadrangle, and each side of p is perpendicular to a face of T, then each side of p is proportional to the area of its corresponding face and, for one direction about p, is directed from the outside to the inside of this face.*

An indication of the proof of this lemma will suffice. Let m be any line in space and M a plane perpendicular to m. Let us project p orthogonally on m and T orthogonally on M. Since the sum of the projections on M of the signed areas of the faces of T is equal to zero, it follows that the sum of the projections on m of the signed segments of p is also zero. But m was taken arbitrarily, whence we see that p must be closed.

The converse is easily proved. Let us be given a line segment, p_1, perpendicu-

lar to face T_1 of T. Now there is one and only one skew-quadrangle $p_1p_2p_3p_4$ such that p_i is perpendicular to T_i. This, with the direct part of the lemma, establishes the converse.

Now, returning to the original problem, let G_a, G_c, G_b, G_d be the feet of the perpendicular from G_1 on planes $C_1B_1D_1$, $A_1B_1D_1$, $A_1C_1D_1$, $A_1C_1B_1$ respectively. Then

$$G_aG_1 \cdot AC = \text{power of } A \text{ with respect to sphere } (G_1C),$$
$$G_cG_1 \cdot CB = \text{power of } C \text{ with respect to sphere } (G_1B),$$
$$G_bG_1 \cdot BD = \text{power of } B \text{ with respect to sphere } (G_1D),$$
$$G_dG_1 \cdot DA = \text{power of } D \text{ with respect to sphere } (G_1A).$$

But, designating the altitudes of $A_1C_1B_1D_1$ from A_1, C_1, B_1, D_1 by h_a, h_c, h_b, h_d respectively, we have

$$G_aG_1 : G_cG_1 : G_bG_1 : G_dG_1 = h_a : h_c : h_b : h_d.$$

And, from the lemma,

$$AC : CB : BD : DA = C_1B_1D_1 : A_1B_1D_1 : A_1C_1D_1 : A_1C_1B_1.$$

Therefore

$$G_aG_1 \cdot AC = G_cG_1 \cdot CB = G_bG_1 \cdot BD = G_dG_1 \cdot DA = k \text{ (vol. of } A_1C_1B_1D_1),$$

where k is some constant of proportionality. This proves our theorem.

Notes: The analogous problem for a triangle in the plane is also true, but is much more easily proved.

The direct part of the lemma and its proof are readily extended to the case of any convex n-hedron T and the resulting skew n-gon p. The converse, however, is no longer true, as is seen by considering skew 6-gons associated with a cube.

Air Flight in Wind

4132. [1944, 475]. *Proposed by T. H. Matthews, McGill University*

If an aircraft travels at a constant airspeed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

Solution by Gordon Pall, McGill University. Let V, W, v be respectively the airspeed of the craft when there is no wind, the wind speed with a constant direction, and the resultant speed of the craft in the forward direction along the closed curve C which it describes; and let τ be the angle from the wind direction to the tangent to C. If $\pi/2 < |\tau| < \pi$ we must have $V > W$. In all cases

(1) $V^2 = W^2 + v^2 - 2Wv\cos\tau$, $v = R + W\cos\tau$, $R = \sqrt{V^2 - W^2\sin^2\tau}$.

The negative sign before R must be dropped since $v \geqq 0$ and, if the negative sign is used, we would have $W \geqq V$. The total time T to traverse C is then

$$(2)\quad T = \int \frac{ds}{v} = \int \frac{R - W\cos\tau}{V^2 - W^2}\,ds = \int \frac{R\,ds}{V^2 - W^2}, \quad \text{since} \quad \int \cos\tau\, ds = 0.$$

Then

$$T = \int \frac{\sqrt{V^2 - W^2\sin^2\tau}}{V^2 - W^2}\,ds > \int \frac{\sqrt{V^2 - W^2}}{V^2 - W^2}\,ds = \int \frac{ds}{\sqrt{V^2 - W^2}},$$

$$T > \int \frac{ds}{V}.$$

This proves the desired result.

Solved also by Murray Barbour, A. M. Glicksman, T. A. Mossman, and G. W. Petrie.

Ring of Integers

4133 [1944, 475]. *Corrected. Proposed by A. L. Putnam, Yale University*

Let a, b, and c be integers with $b \neq 0$, and let d and f be the respective greatest common divisors of a and b and c and b. Then if

$$a \not\equiv \pm d \pmod{b} \text{ and } c \not\equiv \pm f \pmod{b},$$

there is an infinite number of integers k for which the equation

$$ax + bxy + cy = k$$

has no solutions in integers.

I. *Solution by E. P. Starke, Rutgers University.* The given equation may be written

$$(bx + c)(by + a) = bk + ac,$$

and division throughout by df gives

$$(1)\qquad \left(\frac{b}{f}x + \frac{c}{f}\right)\left(\frac{b}{d}y + \frac{a}{d}\right) = \frac{b}{df}k + \frac{a}{d}\cdot\frac{c}{f}.$$

If b is not a multiple of df, this equation has no solution in integers if k is any one of the infinity of integers prime to df, for then all terms are integers except bk/df. On the other hand if b is a multiple of df, b/df and ac/df are coprime because b/f and c/f are coprime, as are b/d and a/d. There are then (Dirichlet's theorem: See Dickson, *Modern Elementary Theory of Numbers*, 1939, pp. 291–305) infinitely many terms in the arithmetic progression, $ac/df + b/df$, $ac/df + 2b/df, \cdots, ac/df + kb/df, \cdots$, which are primes. For each value of k corresponding to one of these primes, one of the factors in the left member of (1) must be ± 1. Hence one of the equations

$$bx + c = \pm f, \qquad by + a = \pm d$$

must be true. Since these are contradicted by the hypothesis, there is no integral solution for such values of k.

II. *Solution by Gordon Pall, McGill University.* We shall prove a more general result from which Putnam's follows:

The function $\phi = bxy+ax+cy$ (where a, b, c are integers, and $b \neq 0$) represents all but a finite number of integers if and only if either

$$a \equiv \pm 1 \pmod{b}, \quad \text{or} \quad c \equiv \pm 1 \pmod{b}, \tag{2}$$

or

$$|b| = 6, \qquad a \equiv \pm 3, \qquad c \equiv \pm 2 \pmod{6}, \tag{3}$$

or vice versa for a and c. When (2) holds, ϕ represents all integers; when (3) holds, ϕ represents all integers save one, namely $-ac/6$.

Proof. Consider the equation $bxy+ax+cy=n$, or

$$(bx + c)(by + a) = bn + ac. \tag{4}$$

Replacing x by $x-h$ and y by $y-k$, we can add arbitrary multiples of b to a and c, while leaving $bn+ac$ unchanged. By altering the signs of x, y, and ϕ, we can secure $b>0$, $a \geqq 0$, $c \geqq 0$. These operations leave $\pm(bn+ac)$ unchanged, and reduce the problem to a form ϕ in which

$$0 \leqq a \leqq \tfrac{1}{2}b, \qquad 0 \leqq c \leqq \tfrac{1}{2}b. \tag{5}$$

If either a or c is 1, ϕ evidently represents all integers. If $a=0$ and $c>1$, then $\phi = y(bx+c)$ cannot represent any prime of the form $bn \pm 1$.

We can thus assume that $a \geqq c \geqq 2$, $b \geqq 5$. If $|x| \geqq 2$ and $|y| \geqq 2$, then $|bxy+ax+cy| \geqq 2a$ by (5). If x and y have the values 0, ± 1, the only values of ϕ which may equal 1 or 2 are $b-a-c$, a, c. Hence if ϕ is to represent 1 and 2, $b-a-c=1$, $c=2$, whence by (5) either

$$b = 5,\ a = c = 2; \quad \text{or} \quad b = 6,\ a = 3,\ c = 2.$$

In the first case ϕ does not represent 3. In the second case,

$$6xy + 3x + 2y + 1 = (3x + 1)(2y + 1),$$

where $3x+1$ represents $\pm 2^h$, and $2y+1$ is any odd number. Hence $6xy+3x+2y$ represents every integer except -1.

It follows that except when (2) or (3) holds, ϕ fails to represent some integer n such that $bn+ac \neq 0$. That is, in view of (4), if $bn+ac = \pm p_1 p_2 \cdots p_r$ as a product of primes, we cannot partition the product into two factors one of which is congruent to c, the other to a, mod b. The same property must hold for $\pm p_1 p_2 \cdots p_r p$, if p is any prime congruent to ± 1 mod b. Hence ϕ fails to represent infinitely many integers n.

We have used here the fact that *there exist infinitely many primes of the form $nb \pm 1$*. This was proved by *elementary* methods by several writers. (For refer-

ences, see Dickson's *History of the Theory of Numbers*, I, pp. 418–419.) It is of course a special case of Dirichlet's theorem on primes in an arithmetic progression, which has however not yet been proved by strictly elementary means.

Gordon Pall found the conditions for ϕ to represent all integers in 1931 (*A class of universal functions*, Bulletin of the American Mathematical Society, 38, 1932, 56–58), but overlooked the fact that $(3x+1)(2y+1)$ did not represent zero.

Solved also by the proposer as corrected.

Editorial Note. The above two solutions gave also proofs that the original statement with $d \not\equiv \pm f \pmod{b}$ is incorrect. Pall gave also a proof, using Dirichlet's theorem, independent of his above proof. The proposer's solution, also using this theorem, pointed out the following context in which the problem occurred:

Suppose R is a commutative ring with a unity and B is a proper ideal of R. In general the residue class $B+ac$ is larger than the set consisting of all products of elements from the two residue classes $B+a$ and $B+c$. It is natural to ask when equality will hold. In the special case of the ring of integers the ideal B has the form (b) where $b \neq 0$, and a necessary condition becomes: The equation

$$(a + by)(c + bx) = ac + kb$$

has integral solutions x and y for every integer k. As is shown in the solution of the problem, this requires solving the equation given in the problem.

Angle Trisection

4134 [1944, 475]. *Proposed by Hüseyin Demir, Columbia University*

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n-\pi/3)(A_i-\pi/3)=(-1)^n2^{-n}$.
2. The limit of the direction of $C_2^nC_3^n$ as $n\to\infty$, is the direction of one of the trisectrices of the angle $(A_2A_3, C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

Solution by Howard Eves, College of Puget Sound. 1. Designating the incenter of $A_1A_2A_3$ by I we have $C_2^1IC_3^1=2C_1^1$. Therefore $A_1+2C_1^1=\pi$. Similarly, $A_i+2C_i^1=\pi$, or

$$(C_i^1 - \pi/3)/(A_i - \pi/3) = -2^{-1}.$$

By the same process

$$(C_i^j - \pi/3)/(C_i^{j-1} - \pi/3) = -2^{-1}, \qquad j = 2, \cdots, n.$$

By multiplication we then get

$$(C_i^n - \pi/3)/(A_i - \pi/3) = (-1)^n 2^{-n}.$$

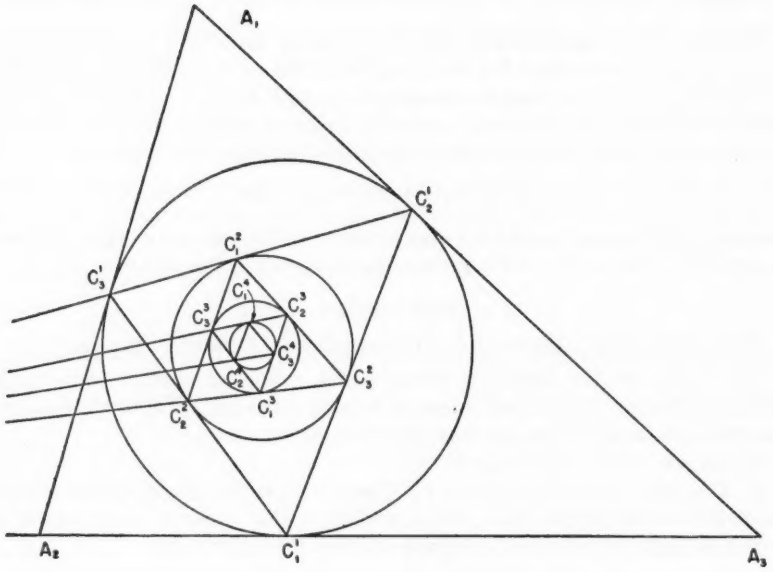
2. Now

$$\lim_{n\to\infty}[\text{angle } (A_2A_3, C_2^nC_3^n)] = \sum_{n=1}^{\infty} \text{angle } (C_1^{2n-2}C_2^{2n-2}, C_1^{2n}C_2^{2n}), \quad \text{where} \quad C_i^0 \equiv A_i;$$

$$= \sum_{1}^{\infty} (C_3^{2n-1} - C_1^{2n})$$

$$= \sum_{1}^{\infty} [(C_3^{2n-1} - \pi/3) - (C_1^{2n} - \pi/3)]$$

$$= \sum_{1}^{\infty} [-2^{-2n+1}(A_3 - \pi/3) - 2^{-2n}(A_1 - \pi/3)],$$



by the relations of part 1 above. The infinite sum on the right reduces to

$$\sum_{1}^{\infty} [\pi - A_1 - 2A_3]2^{-2n} = (A_2 - A_3)\sum_{1}^{\infty} 2^{-2n} = \tfrac{1}{3}(A_2 - A_3) = \tfrac{2}{3}\left(\frac{A_2 - A_3}{2}\right);$$

$$= (\tfrac{2}{3})[\tfrac{1}{2}(A_2 - \pi/3) - \tfrac{1}{2}(A_3 - \pi/3)]$$

$$= (\tfrac{2}{3})[-(C_2^1 - \pi/3) + (C_3^1 - \pi/3)]$$

$$= (\tfrac{2}{3})(C_3^1 - C_2^1) = (\tfrac{2}{3}) \text{ angle } (A_2A_3, C_2^1C_3^1).$$

This establishes part 2. The method suggested here for asymptotically obtaining one of the angle trisectors of a given acute angle $(A_2A_3, C_2^1C_3^1)$ is apparent. It is

needless to say, however, that there are many better euclidean asymptotic constructions for trisecting an angle.

Solved also by the proposer.

Editorial Note. Below are some references to this MONTHLY regarding approximate methods of angle trisection with limits for the error:

1932, 478, *Angle Division*, article by E. C. Kennedy; 2972 [1925, 95]; 3114 [1925, 483]; 3522 [1933, 303]; 3563 [1934, 113]; The method of Pappus using conics 3490 [1932, 243].

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Assistant Professors H. G. Apostle and A. E. Basch of the University of Rochester and Rensselaer Polytechnic Institute, respectively, have been appointed to assistant professorships at Amherst College.

Associate Beulah M. Armstrong of the University of Illinois has been promoted to an assistant professorship.

Assistant Professor J. H. Butchart of Grinnell College, Grinnell, Iowa, has been appointed to a professorship at Arizona State Teachers College.

Dr. F. A. Butter, Jr., has been appointed acting assistant professor of mathematics at Stanford University.

Professor A. G. Clark and Assistant Professor M. L. Madison have returned to Colorado State College of Agricultural and Mechanical Arts.

Assistant Professor D. B. DeLury of the University of Toronto has been appointed to an associate professorship at Virginia Polytechnic Institute.

Assistant Professor R. P. Dilworth of the California Institute of Technology has been promoted to an associate professorship.

Dr. T. C. Doyle of Stanford University has been appointed to an assistant professorship at Dartmouth College.

Assistant Professor Samuel Eilenberg of the University of Michigan has been promoted to an associate professorship. He is, at present, visiting lecturer at Princeton University.

W. C. Griffith of DePauw University has been appointed to a professorship at Marion Institute, Marion, Alabama.

J. R. Hanna of the University of Wichita has been promoted to an assistant professorship.

Assistant Professor J. D. Hill of Michigan State College has been promoted to an associate professorship.

Assistant Professor K. D. Kelly of Fenn College, Cleveland, Ohio, has been promoted to an associate professorship.

Dr. Lois Kiefer of New Mexico State Teachers College has been appointed to an acting assistant professorship at the University of Tennessee.

Professor Lincoln LaPaz of Ohio State University has been appointed head of the department of mathematics at the University of New Mexico.

Associate Professor Eugene Lukacs of Berea College has been appointed to a professorship at Our Lady of Cincinnati College.

Associate Professor M. S. MacPhail of Acadia University, Wolfville, Nova Scotia, has been promoted to a professorship.

Dr. V. O. McBrien of the College of the Holy Cross has been promoted to an assistant professorship.

Professor Edith J. McKissock of Youngstown College, Youngstown, Ohio, has been appointed assistant dean of women at the University of Cincinnati.

Professor A. E. Meder, secretary of Rutgers University, has been appointed dean of administration.

Dr. N. S. Mendelsohn has been appointed lecturer at Queen's University, Kingston, Ontario, Canada.

Associate Professor R. R. Middlemiss of Washington University has been promoted to a professorship.

Dr. H. E. Nelson of Gustavus Adolphus College, St. Peter, Minnesota, has been appointed associate professor of mathematics and head of the department.

Associate Professor R. M. Pinkerton of the Agricultural and Mechanical College of Texas has been appointed professor of mathematics and acting head of the department.

Dr. A. R. Poole has been appointed to an assistant professorship at Montana State College.

Fred Robertson of Iowa State College has been promoted to an assistant professorship.

Dr. L. D. Rodabaugh of Oberlin College has been appointed statistician in the Bureau of the Census in Washington.

Dr. Peter Scherk of the University of Saskatchewan has been promoted to an assistant professorship.

Associate Professor Samuel Selby of the University of Akron has been appointed professor of mathematics and head of the department.

Dr. D. R. Shreve has been appointed to an associate professorship at the University of Tulsa.

Dr. W. C. Taylor of the University of Cincinnati has been promoted to an assistant professorship.

Assistant Professor C. C. Torrance of Case School of Applied Science has been promoted to an associate professorship.

Dr. B. R. Ullsvik of the State Teachers College in Eau Claire, Wisconsin, has been appointed to an associate professorship at Illinois State Normal University, Normal, Illinois.

Dr. Mary C. Vanhorn of Trinity College, Washington, D. C., has been promoted to an assistant professorship.

Dr. G. C. Vedova has been appointed professor of mathematics and head of the department at Pennsylvania Military College, Chester, Pennsylvania.

Assistant Professor G. L. Walker of the University of Delaware has been appointed to an assistant professorship at Temple University.

Dr. H. S. Wall of the Illinois Institute of Technology has been appointed to a professorship.

Dr. S. E. Warschawski has been appointed professor of mathematics and mechanics at the University of Minnesota.

The following appointments to instructorships are announced:
Everett Junior College, Everett, Washington: G. H. Van Arkel
North Carolina State College: G. C. Watson
Southern College of Optometry, Memphis, Tennessee: R. L. Coker
University of California: Dr. O. G. Owens
University of Chicago: Dr. Irving Kaplansky, Dr. William Karush
University of Illinois: Corinne Hattan
University of Minneaota: W. D. Munro

Professor Stefan Banach, according to a report from the Polish Press Agency died in Lvov at the age of fifty-three.

N. Durairajan, Executive Engineer in Mylapore, Madras, India, died July 15, 1945.

Dr. Willis Whited of Harrisburg, Pennsylvania, died April 28, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSITY PROGRAMS FOR EX-SERVICE PERSONNEL IN CANADA

F. S. NOWLAN, University of British Columbia

This paper is divided into two parts. The first is an abridgment of a talk that the author gave in June, in Montreal, before the Canadian Mathematical Congress. The second part represents an attempt to bring the Canadian picture up to date. Although educational conditions in the two countries are different, frequent reference is made to the American situation. It is believed that these comparisons are worthwhile.

Part I

The world situation has undergone great change since I was asked to undertake this survey. The shooting war in Europe is over and the next few months will see thousands of young men and women return to civilian life. Many of these left the high schools and universities to enter the armed services. For various reasons they wish to renew their schooling. They possibly realize that the brightest hope for a peaceful world lies in a new and broader outlook on life. To them, education offers the one opportunity to gain this outlook and to contribute to the building of an enlightened and vigorous Canada in a world, we hope, of finer ideals.

Numbers of these young people have witnessed the power of science. Their imaginations have been stimulated and they look forward to creative work in science or engineering. Some have seen and experienced the healing magic of medicine and they feel the urge to embark upon its study. There are other considerations, but whatever the urge, carefully conducted surveys have shown that thousands of young airmen, sailors, and soldiers propose to continue their studies and soon will be crowding our universities.

This influx will entail increased responsibilities for us. However, these young men and women have never failed us and they have the right to expect our best efforts in this emergency. Nevertheless, it is fair to add that we can not assume the entire responsibility in connection with the problems that arise. There are federal obligations, and federal aid should be given beyond the payment of gratuities, which, it is true, have been provided on a generous scale.

The terms of the Post Discharge Re-establishment Order-in-Council, P.C. 5210, provide for a maintenance grant on a graduated scale and the payment

of the usual fixed university fees,* exclusive of prescribed textbooks, to scholastically qualified and honorably discharged ex-service personnel for as many months as the individual has been in the Service. The monthly maintenance grant varies from $60 for an unmarried man or woman, with no dependents or disabilities, and $80 for a man and wife under similar conditions, to a hypothetical maximum of $197 in the case of a man with total disability (such as blindness), a wife and six children. An additional grant is made for each child. The allowance for each of the first two children is $12 a month, $10 is allowed for the third child, *etc.* In addition, $15 a month is allowed for a dependent parent.

The grants are designed to relieve the recipient from financial worry and so to enable him to devote his entire attention to his studies. They are on a somewhat more generous scale to the individual than the corresponding sums provided in the U.S.A. under the terms of Public Law 346, the G.I. Bill of Rights. The difference is accentuated by the fact that the cost of living in Canada is less than in the U.S.A. The American ex-service man receives a monthly maintenance grant of $50 or $75, depending upon his single or married status and, in addition, the payment of university fees, including textbooks and supplies, up to $500 a year. This sum may be increased in the case of universities which normally charge a higher tuition fee.

The conditions that govern Canadian grants to veterans are set forth in two pamphlets issued by the Department of Pensions and National Health. These are entitled *Back to Civil Life* and *Principles Governing Training for Rehabilitation.* For purpose of comparison, the educational provisions of the G.I. Bill of Rights can be found in this MONTHLY for October, 1944. Further information may be found in the numerous Bulletins on Higher Education and National Defense as issued by the American Council on Education, Washington, D. C.

The best available information indicates that 35,000, or more, ex-service personnel will seek admission or re-admission to Canadian universities within the next few years. What this implies is obvious when one considers that the total pre-war registration in our universities was approximately the same number. The surveys indicate that courses in engineering will be in greatest demand. Next in popularity is commerce and business, followed in order by arts, education, and medicine.

The magnitude of the administrative problem is appreciated when one realizes that the fees of a Canadian university student pay only about 40% of the institutional cost of his education. The question of buildings and equipment in universities that are already overcrowded and understaffed is extremely serious. It is even conceivable that unless some solution of this problem is reached, attendance at an institution such as the University of British Columbia, which is

* The regulations have been modified since this paper was given in Montreal. They now provide for an additional payment *to the university*, for each ex-service man, of $150 per university year, and a proportional sum if the student takes summer work. An additional sum of $5 weekly is granted a married student who supports two establishments. This provides for cases where a student's wife is not with him at the university, or where the student has dependent parents who do not live with him.

without dormitories, may be denied many veterans due to the difficulty in finding shelter.

A detailed discussion of Canadian university problems related to the training of veterans can be found in the Report of the National Conference of Canadian Universities on Post-War Problems, as adopted at their meeting at McMaster University in June, 1944. This Report is well worth reading. It is constructive, but non-controversial. In this latter respect it is in marked contrast to published statements from individual American educators in regard to the application of the G.I. Bill of Rights. However, these diverse points of view help in forming a well-balanced picture of the situation. I have particular reference to the fears expressed by President Hutchins of the University of Chicago* regarding the educational provisions of the G.I. Bill of Rights. Also, President Conant of Harvard University in his "President's Report" for 1943–44, expresses misgivings.

I feel that to some extent we should take issue with these gentlemen. They seem to fear a general lowering of standards. To quote President Hutchins: "A serious dilution of an already diluted educational system is in prospect." It occurs to me that the influx of new students may give us both the excuse and opportunity for raising standards. On the other hand, it would admittedly be unfair to expect higher standards of service men than we demand of civilian students, and so long as we tolerate play-boys and play-girls in our universities, I feel that we may dismiss our fears in regard to the veteran letting us down scholastically. So far, the experience both in Canada and the U.S.A. has been that the veteran is a serious-minded, hard worker, who attains better than average standing.

I recently sent a questionnaire to the various Canadian colleges and universities in the hope that I might get a more intimate picture of the problems which confront us as teachers of mathematics. The overall picture that I get is as follows:

1. There probably will not be as many ex-service men in university attendance as indicated by official surveys. Nevertheless, Toronto considers the possibility of 10,000 ex-service students within the next few years. Mt. Allison is preparing for a possible 50% increase in registration and the University of New Brunswick anticipates a similar increase. In general, the Western Universities hesitate to speculate upon the increase but this year the University of British Columbia had 2908 students during the regular session. It is expected that at least 3500 will register in the fall.

2. As a general rule, refresher courses will not be given and the feeling is that they are not required. Special summer courses will take care of the needs.

3. There is a keen realization of the importance of counselors, with academic and war experience, to advise and guide the service men. But, so far as possible, these men should be kept from feeling that they are receiving special attention.

4. Dean Beatty of the University of Toronto expressed the general feeling in

* *Cf.* The State of the University, October 19, 1944.

his reply: "I believe that as far as possible the instruction will be entrusted to the older and more experienced members of the staff, men who will appreciate what the boys have come through and who will understand where they are likely to find difficulties. The boys will need careful and sympathetic attention."

5. It is felt that the classes should be kept reasonably small. A number of educators favor classes of not more than 30. The largest figure reported comes from Toronto which sets 50 as a maximum.

6. It is felt that there is no likelihood, nor danger, of a lowering of standards and there would be opposition to any step in that direction. On the other hand, more emphasis will be placed upon the practical applications of mathematics.

The impression in the universities that have instructed groups of ex-service men during the past year is that in their studies these men more than keep up their end as compared with civilian students. Our U.B.C. group consisted mainly of airmen and at one time included six men with the D.F.C. Two heads of departments, who instructed these men, informed me that they worked harder than other students and, in fact, were at a higher level of mentality. Our mathematics instructors report that these students wish to work all the problems of the text whereas the average civilian student appears to feel in honor bound to work as few as possible.

These observations are in keeping with my experience last summer when I instructed in the V-12 program at Notre Dame University. The students were navy men and marines, about half of each. A number had seen action in the South Pacific, and I felt that they did the best work in the group. The men were of more than average ability. They were serious minded and among the hardest workers that I have met in my teaching experience.

It is my feeling that the average civilian student is so imbued with the notion of play that as a result we have formed an entirely false idea of how much can be required of him. The ex-service men are serious and have habits of work and application. Through the force of example they can do our universities a real service.

I recently made a study of freshman examination papers in mathematics, as given in our various colleges and universities. There appears to be a considerable divergence in standards and this, insofar as service men are concerned, has undesirable features. It means that a service man might fail in more than two subjects and so, according to the terms of the Act, lose his right to further instruction. Nevertheless, in knowledge and ability he might be superior to a student who had made a fair showing in a university with lower standards. Possibly something of this sort was in President Conant's mind when he made the Report to which we referred. The Congress might well take this matter under advisement.

I have brought to your attention some of the difficulties which face us in the post-war period and for which we have no ready-made solution. However, we should not forget, nor should we let others forget, that these difficulties are by-products of the war and as such it is fair to expect and demand increased

federal aid for their solution. Furthermore, the universities should be given priority in matters of new buildings and equipment. A country that has performed miracles of production will scarcely accept defeat when it comes to implementing plans for post-war reconstruction.

Finally, I would like to see this Congress go on record as recommending that the educational provisions of P.C. 5210 be extended to apply* in another generation to children of those men who have made the supreme sacrifice or who may have become totally incapacitated. Also, among the veterans are older men who are not personally interested in university attendance. It should be possible, under certain conditions, for such a person to transfer his right to an educational grant to a son or daughter. By such an arrangement, we could repay a portion of the debt that we owe these men, and we may be sure that an investment in education pays the best dividends.

Part II

This is a factual Report,—an attempt to picture the unprecedented situation in Canadian universities at the opening of the fall term, 1945–46. These universities, with few exceptions, have experienced a 40% to 90% increase in enrollment. This is due to the influx of ex-service men who are either entering upon or continuing their university education. This increased attendance has entailed some problems of a physical nature and others that relate to scholarship. Also, as is natural, different universities, as well as different departments within the same university, have varied in their anticipation of the problems as well as in their efforts to arrive at solutions.

In general, like conditions of overcrowding, insufficient staff, and inadequate student accommodations prevail in all Canadian universities. The overcrowding is most extreme in first and second year classes in arts and engineering and in courses in mathematics, chemistry, and physics. Since this has been written for a mathematical journal, the picture is restricted in its pedagogical aspects to conditions that affect instruction in that subject. Although reference is made to a number of Canadian universities, the University of New Brunswick, (U.N.B.), has been selected as typical of the eastern universities, and the University of British Columbia, (U.B.C.), to represent the West. At U.B.C., we have a 94% increase in enrollment, the present registration being 5,650, and overcrowding is extreme. Only those who have lived through a Times Square subway crush can appreciate conditions in the corridors of the Arts Building at U.B.C. at the intermission between classes. The subway, however, has the advantage of guards who propel laggards along. That is a device that to date we have overlooked.

In viewing the picture, we should remember that, traditionally, mathematics is a required subject of study for first year students in arts at Canadian universi-

* Resolutions favoring this, and the suggestion that follows, were adopted by the Congress and forwarded to the appropriate authorities.

ties. (Its study is said to have cultural value.) In addition, the subject is required in several years of engineering. It follows that, with the possible exception of English, more students take mathematics than any other subject in the curricula, and that, with the exception of science courses that involve laboratory work, mathematics is most affected by any sudden increase in registration.

University of New Brunswick. This is one of the smaller universities. There are 300 ex-service men in attendance this year, representing a 75% increase over last year's total registration. The department of mathematics gives 2,320 student-hours of instruction per week as compared with 1,180 last year. The mathematics staff has been increased from two to three professors. In addition, 9 senior students grade papers in mathematics, each for two hours a week. There are no special arrangements for helping students who may have difficulty with their work. Instructors meet first and second year classes in groups that range in numbers from 60 to 98. Renovation of existing buildings has provided the necessary classroom and laboratory space. A house to house survey of Fredericton found living quarters for all but 50 men. These have been provided with temporary quarters in the gymnasium and they do their studying in classrooms and the library. It is expected that 150 additional ex-service men will register and begin work in January.

Acadia University has found it necessary to limit civilian admissions.

Mt. Allison University has instituted a unit term system which has proved very helpful. The university provides assistance for veterans who need it, as well as regular tutorial periods for students in first and second year mathematics.

The University of Toronto has experienced a 50% increase in attendance and they expect a large additional enrollment in January. They will then take over 70,000 square feet of floor space from the Ajax defense plant which is located 18 miles from the University proper. This space will be utilized for housing and training some 1500 students in first year engineering who will begin their courses at that date.

Queens University expected, in early September, that a third of their students would be ex-service men.

The University of Western Ontario, as of October 2, had registered 800 freshmen, which is more than their total registration of arts students last year. The increase was due to ex-service men.

The University of Saskatchewan has approximately 1,200 veterans, with another large group expected in January.

The University of Alberta has been obliged to set a quota upon enrollment in its professional schools.

The University of British Columbia. The present plant of U.B.C. was opened in the fall of 1925 and was built to accommodate 1,500 students. In spite of a steady and rapid growth to last year's total of 2,908 students, there has been no extension of initial classroom space. It is true that last spring the provincial legislature voted $5,000,000 for university expansion, but federal restrictions and priorities have prevented any building. This was the situation when it was

realized last spring that the fall registration would be phenomenal. The problems that were faced were of two kinds, the physical (classrooms, equipment, and student accommodations), and the scholastic.

The Physical Problem. This problem entailed much labor in the way of negotiations and supervision. It owes its successful solution to painstaking work on the part of the administration and certain faculty members, combined with fine cooperation from the military authorities. As a consequence:

(a) The university obtained an army camp on Acadia Road, a half mile from the campus, as a gift from the Army.

(b) The Army loaned the university most of the buildings at the Point Grey Fortress, a coast artillery station 200 yards from the campus.

(c) The university purchased from the War Assets Corporation a heavy AA camp on Chancellor Boulevard, about two miles from the campus.

(d) The university obtained on indefinite loan from the R.C.A.F. three buildings at Jericho Beach, about four miles from the campus.

In addition to the foregoing, the university has obtained from the Army eleven buildings which were part of an anti-aircraft camp on Lulu Island. These buildings will be brought to the campus and used for living accommodations. The buildings will be ready by January, 1946, when, at least, 1,200 additional students (ex-service men) are expected. Furthermore, the Army and R.C.A.F. have provided, on loan, equipment for two kitchens as well as beds, blankets, and equipment for the dormitories.

The Scholastic Problem. Last year, the mathematics department at U.B.C. consisted of six full-time instructors and one part-time instructor. It has been increased this year by the return of a staff member who was on leave and by the appointment of five instructors, three young men and two women. In addition, there are seven student assistants (tutors) and four readers of class exercises.

Last year the mathematics department gave 6,230 student-hours of instruction per week. This year it gives 12,685 hours. The class sections are large, several ranging in number from 250 to 290. There are 1,730 first year students in arts, and they take their mathematics in eighteen sections. Of these, fourteen are between 64 and 85 in number, while four exceed 100. One section is around 260 in number. The instructor in charge of this section gives 3,010 student-hours of instruction a week.

No refresher courses are offered, but certain rooms are designated as study rooms for several hours each day, and are in charge of advanced students who give help if requested. Also, members of the Students' Council have offered to provide assistance as far as they are able. This will be provided without charge.

It has been indicated that this report is objective and, as such, personal opinions are out of place and are not expressed. It may, however, be observed that many of the ex-service men have been away from mathematics for periods ranging from two to seven, or more, years. It follows that personal attention is imperative if they are to have a reasonable chance of success in their work.

In closing, the writer apologizes to Presidents Conant and Hutchins.

THE AVAILABILITY OF METEOROLOGISTS AS INSTRUCTORS

The American Meteorological Society through its President, Professor C.-G. Rossby of the University of Chicago, wishes to call to the attention of American colleges that among the large number of carefully selected college students who were given professional academic training in meteorology and have served as professional meteorologists (weather officers) with the Army Air Forces and the Navy Aerological Service during the war, there are a number who before their entry into the meteorological field had done a considerable amount of graduate work in mathematics and who now might become available for teaching positions in the fields of mathematics and meteorology.

Most of our colleges are now offering nonprofessional courses in meteorology to meet the needs of students in geography, geology and in agricultural subjects. In the post-war period the interest in flying is likely to extend and thus a need will arise for nonprofessional courses in aeronautical meteorology, designed to meet the needs of liberal arts students who, as private flyers, might wish a certain amount of information about the atmosphere. The needs listed above are, however, in most cases so limited as to preclude the possibility of the employment of properly qualified professional meteorologists as full-time instructors and it has thus become standard practice to have these nonprofessional courses in meteorology taught by members of other departments. The return of many of the professionally trained service meteorologists offers an opportunity for our colleges to obtain the service of instructors who are capable of doing professional work in a mathematics department and at the same time are qualified as experienced and trained professional meteorologists to present stimulating nonprofessional courses in meteorology on a somewhat higher plane than may have been possible in the past.

In some cases the mathematicians who were brought into meteorology during the war acquired useful experience in certain phases of applied mathematics, particularly theoretical hydrodynamics, through research activities in theoretical meteorology.

The American Meteorological Society is anxious to aid in the development of a program which will permit the country to salvage some of the great forward strides made in meteorology during the war. As part of this program, the Society maintains an extensive file on the academic education, training and professional meteorological experience of the meteorologists who served with our Armed Forces during the war. The Society will be glad to assist any college or college department which might be interested in obtaining the services of properly qualified meteorologists, by providing such organizations with names of suitable candidates together with information concerning their academic background, experience and general suitability for academic work. Inquiries should be directed to the Executive Secretary, American Meteorological Society, 5727 University Avenue, Chicago 37, Illinois.

MEXICAN MATHEMATICS

RUFUS OLDENBURGER, Illinois Institute of Technology

The second National Mathematical Congress of Mexico in three years was held at Guadalajara in the State of Jalisco, May 28 to June 2, 1945. Over two hundred professional leaders attended including Señor don Jaime Torres Bodet, Secretary of Education in President Camacho's cabinet. Senor Torres gave one of the main addresses. The congress was dedicated to the late George D. Birkhoff whose lectures at the National University in 1942 and 1943 inspired much research activity on his theory of gravitation. The United States was represented by Professors Nelson Dunford, Solomon Lefschetz, Francis D. Murnaghan, Norbert Wiener, and Rufus Oldenburger. Professors Lefschetz and Wiener spoke on Birkhoff and his work.

Several papers were presented to the congress by the new group of young pure and applied mathematicians, including Professors Alberto Barajas and Roberto Vásquez of the Institute of Mathematics of the National University, and Professors Antonio Romero Juárez, Nabor Carrillo Flores and Carlos Graef Fernández of the Institute of Physics of this university. Of the men just named Professor Vásquez is the only one specializing in pure mathematics, his interest being in analysis. The formerly excessive teaching loads of these men have recently been reduced to as little as twelve hours a week, largely through the influence of Professor Ricardo Monges López, director of the faculty of sciences of the National University, whose kindness and fatherly encouragement mean a great deal to the scientists of Mexico.

The participants in the congress were entertained by the Governor of Jalisco at a theater performance of dancers, singers and instrumentalists, and at a banquet at the French Club. The University of Guadalajara took the congress to Lake Chapala for a day. In many ways the congress was honored by the government, receiving its financial support.

Although the active research mathematicians in Mexico can be counted on the fingers of two hands, the work of these men receives considerable support from people in other fields on a scale not attained in the United States. Without the attendance of these other men the meetings of the congress would have been very small. Mathematical productivity is being encouraged by the official journal, "Boletín de la Sociedad Matemática," of the Mexican Mathematical Society which has about one hundred and forty members. The president of this society is Dr. Alfonso Nápoles Gándara, director of the Institute of Mathematics of the National University.

At the present time, post-college courses in mathematics are given only by the National University in Mexico City. The encouragement of the government and private sources promises a rapid and extensive expansion of mathematical interest in Mexico, an interest stimulated by the commission of President Avila Camacho for promoting and coordinating scientific investigation.

MARINE CORPS CRITICIZES THE TEACHING OF MATHEMATICS

Lt. Col. Thomas B. Hughes, in charge of the Marine Corps training section, recently discussed the training program of the Marine Corps before the Connecticut School Board Association. His address, as reported by Albert R. Prince in the U. S. Education News, was critical of the instruction given to students in the United States in English, mathematics, foreign languages, and citizenship. Col. Hughes pointed out that too large a percentage of men coming to the Corps had been poor in mathematics, in English, and in citizenship training.

Much of the trouble encountered by the training section of the Marine Corps appeared to be caused by fundamental difficulties in the use of English. In too many cases, special training was necessary to bring trainees to the fourth grade level in their use of English. Moreover, "some who had had two years of college were so weak in English that they could not be approved for officer training. One such candidate made 60 mistakes in a one-page letter."

Col. Hughes recommended the adoption of a well rounded high school program with considerable emphasis upon communication rather than literature in the English courses, and with a minimum requirement of two years of mathematics. He also urged that more attention be given to the foreign languages, with new emphasis upon Russian, Spanish, and Portuguese.

THE MATHEMATICAL ASSOCIATION OF AMERICA

CALENDER OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
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NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
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ROCKY MOUNTAIN
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SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
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Edited by OTTO DUNKEL, Washington University, ORRIN FRINK, JR., Pennsylvania State College, H. S. M. COXETER, University of Toronto, and HOWARD EVES, College of Puget Sound.

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